Proceedings of The National Conference
On Undergraduate Research (NCUR) 2018
University of Central Oklahoma
Edmond, Oklahoma
April 5-7, 2018

Gravity After Einstein: Developing a Model to Analyze the Distance to the Moon

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Abstract

Experimentally testing post-Einstein theories of gravity is imperative for bridging the gap between quantum mechanics and general relativity. Precisely measuring the distance from the Earth to the Moon through lunar laser ranging (LLR) is a leading way to test these new gravitational theories, and millimeter precision LLR is currently conducted by the Apache Point Lunar Laser-ranging Operation (APOLLO). This project proposes a computational model of the Solar System to which APOLLO's experimental data can eventually be fit. Making this computational model requires a variety of considerations, including choice of numerical integration techniques, model simplifications to ensure early editions of the model are running correctly and efficiently, and fitting methods to find an ideal set of parameters and gravitational theories that work best with a given set of data. In the process of creating this model, comparisons are made to the Planetary Ephemeris Program (PEP), the only open source model of the Solar System known to produce LLR observables, accurate to a centimeter. In addition to serving as a platform to learn about the Solar System and lunar range modeling techniques, this work opens the possibility to expand the scope of relativistic gravity tests beyond the current capabilities of PEP. The current model is precise to 40 microns for simple models of the Solar System. In the future, the model will be extended to introduce more complex factors that influence the lunar orbit in order to achieve agreement within a millimeter of experimental APOLLO results and to assess post-Einstein relativistic gravity theories.

Keywords: Lunar Laser Ranging, Computational Model, General Relativity

1. Introduction

Lunar laser ranging (LLR) is a technique of measuring the launch and receive times of a series of electromagnetic laser signals from an observatory on Earth to one of five corner cube retroreflectors on the Moon and back. These timing measurements translate to range estimations, allowing the mapping of the lunar orbit over time. The corner cube reflectors that reflect the laser directly back to its source were installed first by astronauts on the Apollo 11 mission and subsequently by the Apollo 14 and 15 missions, as well as by two unmanned Soviet rovers.

Even though Einstein's theory of general relativity is fully consistent with existing experimental constraints, there are theoretical reasons to suggest that it might not be complete. General relativity is incompatible with quantum theory, a theory that describes the nature of matter and energy at the atomic and sub-atomic level. Of the four known fundamental forces of nature, general relativity has the weakest constraints. This, on top of the unification of all forces but gravity, suggests that our understanding of the gravitational interaction needs to be revised.

LLR data plays a key role in informing our understanding of gravitational interactions. Its unprecedented levels of precision offer a way to discriminate between different gravitational theories at an exceptional level of precision.

Since eight days after the completion of the Apollo 11 mission, scientists have been collecting lunar ranging measurements. To date, an archive of LLR data spanning 49 years is available. The first accurate laser ranging measurement was done on August 1, 1969, at the Lick Observatory using a 3.0-meter telescope.[1] Currently, the Apache Point Observatory Lunar Laser-ranging Operation (APOLLO) at the Apache Point Observatory in southern New Mexico uses a 3.5-meter telescope to measure this transit time with an accuracy of a few picoseconds. This translates to an accuracy of a millimeter in the measured lunar distance.

1.1 The Need For A Computational Model

In order to extract gravitational physics constraints from LLR data, it is necessary to have a computational model that simulates an understanding of gravitational interactions. In constructing this model, all factors that affect the lunar orbit should be taken into account. These factors include the gravitational influences of the Earth, the Sun, other planets in the solar system, and the moons of those planets. Table 1 shows the magnitude of the influence of the solar system planets and Pluto on the lunar orbit. Additionally, lunar librations, surface deformities of both the Earth and the Moon, and the precession of the Earth, which is the movement of the Earth's axis of rotation, should be taken into consideration. Since such a system is too complicated to solve analytically, a computational solution is necessary.

Table 1. The influence of solar system planets (other than the Earth and the Sun), and Pluto on the lunar orbit.

Perturbing Object	Venus	Jupiter	Mars	Mercury	Saturn	Neptune	Uranus	Pluto
Lunar Perturbation (mm)	260000	26000	11000	2000	900	15	14	0.004

There are several existing LLR models including those housed at the Harvard-Smithsonian Center for Astrophysics (Planetary Ephemeris Program (PEP)), at the Jet Propulsion Laboratory (Jet Propulsion Laboratory Development Ephemeris (JPL DE)), and at the Leibniz University in Germany.[2,3,4] Of these models, only PEP is open source.

Over the years, there have been improvements in the experimentally measured distance due to improved telescopes, enhanced timing electronics, and specifically with APOLLO, a particularly large photon return rate. However, there exists a non-zero disparity between the precision of the experimentally measured distance and its model. Figure 1 shows the change over time of the root-mean-square (RMS) residuals between the data and model.

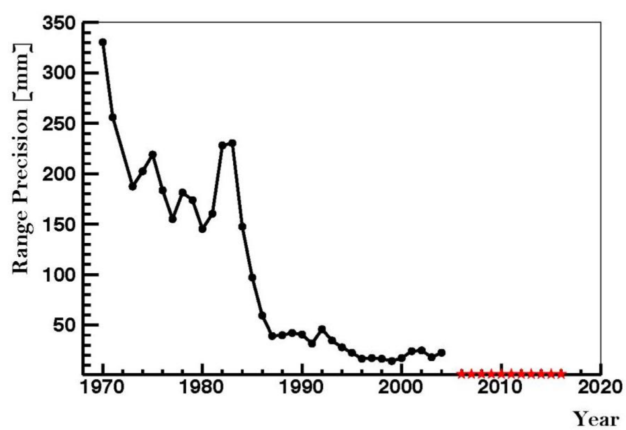

Figure 1: Evolution of the RMS of the residuals between the measured and modeled distance. The residual RMS is due to errors in both the data and model. Each point on the graph is the RMS of the residuals for one year.

1.2 Planetary Ephemeris Program (PEP)

Designed in the 1960s, PEP is written in Fortran 77. Due to the code structure, incorporating relevant physical discoveries made after its construction, such as the presence of dark matter and dark energy, into PEP is a difficult task. Additionally, PEP is accurate at the ~1 cm level. This means it is not able to take full advantage of the millimeter precision of the experimentally measured ranging distance. Furthermore, PEP also has an arcane interface making it unfit for use in an undergraduate context.

As a way to learn more about ephemeris modeling, and with the goal of creating a flexible analysis framework capable of exploring new ideas in gravitational physics, we have begun the development of a solar system ephemeris with a more modern and user-friendly interface. Following PEP, this model has the possibility of being open source.

2. Phases of Creating a Computational Model

2.1. A Simple System

The model presented here was developed progressively and remains a work in progress. The initial model was a simple three-body system that only modeled the motions of the Earth, the Moon, and the Sun. It used Newtonian gravity, and all three bodies were treated as perfect spheres, meaning they could be simulated as point particles. The purposes of this simple system were to develop a precise, accurate integrator and a working fitting technique. The former was tested by comparing the model's results to a PEP run with the same modeling techniques of Newtonian gravity and three perfectly spherical bodies.

2.1.1. integrators

The initial integrator used was the Fourth Order Runge-Kutta Method.[5] This method is essentially a refined Euler method. It evaluates a derivative at an initial point, twice at midpoints, and once at the end of an interval. It then takes a weighted sum of the results to better approximate a final solution. This is shown in figure 2. However, for this model, it became apparent that Fourth Order Runge-Kutta could not achieve the desired levels of accuracy in a reasonable amount of time. Instead, the integrator was changed to the Bulirsch-Stoer Method.[6] This integrator uses an adaptive step size and error control through polynomial extrapolation. Although much more advanced in its technique, Bulirsch-Stoer is more efficient than the Fourth Order Runge-Kutta Method. For this simple three-body system, a 1-meter level agreement on the Earth-Moon distance with PEP over 6000 days is achievable in only 90 seconds with Bulirsch-Stoer. In comparison, such an agreement with the Fourth Order Runge-Kutta method took 250 seconds.

$$
\begin{aligned}
k_1 &= hf'(x, f(x)) \\
k_2 &= hf'(x + \frac{1}{2}h, f(x) + \frac{1}{2}k_1) \\
k_3 &= hf'(x + \frac{1}{2}h, f(x) + \frac{1}{2}k_2) \\
k_4 &= hf'(x + h, f(x) + k_3) \\
f(x+h) &= f(x) + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4
\end{aligned}
$$

Figure 2: The Fourth-Order Runge-Kutta Method. The function being solved for is $f(x)$, and its derivative is $f'(x)$. The step size is given by h.

2.1.2. precision

In order to properly utilize APOLLO's millimeter-accurate lunar laser ranging data, further levels of precision were explored. Because Python uses 64-bit floats, its default precision is limited. In comparison, PEP uses 128-bit quadruple-precision from Fortran 77. To reach a precision closer to PEP, methods of extended precision in Python were explored. First, NumPy's `long double` data type was implemented.[7] This `long double` data type is from most C compilers, and on x86 machines, it is represented with 80 bits. This provides a few more decimal digits of precision, and this extra precision is evident in 6000-day comparisons with PEP. For example, such a comparison using NumPy's `long double` data type was able to achieve agreement with PEP (as defined above) to 1 millimeter.

The `mpmath` package in Python was also explored.[8] This package allows for arbitrary precision, meaning that using `mpmath` would allow for the model's precision to be adjusted depending on the needed accuracy of the result. Additionally, `mpmath` is compatible with NumPy, so `mpmath` was easily implementable. Once it was implemented, we were able to find that 21 decimal digits of precision gave the least disagreement with PEP, as shown in figure 3. Although PEP's 128-bit floating point representation corresponds to about 30 decimal digits of precision, PEP's output only prints 19 digits. Accounting for additional accuracy needed for the integrator, it makes sense that comparisons between this model and PEP's output would reach a minimum slightly above PEP's number of printed digits. However, this simulation in `mpmath` takes much more time. For 20 precise digits, a 6000-day simulation took about 50 minutes, whereas simulations with the `long double` data type take 2 minutes. In later versions of the model, varying data types were used depending on the wanted precision of the final result.

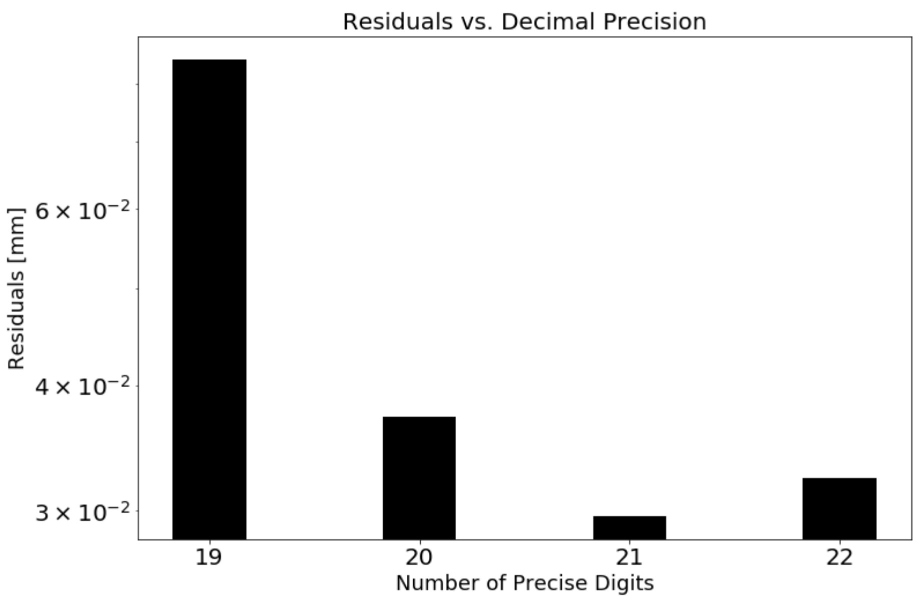

Figure 3: The maximum disagreement between PEP and this three-body model over 6000 days in millimeters, plotted as a function of the number of precise decimal digits in the newly developed simulation.

2.1.3. results

On the macro level, this simple three-body model performed well. In the distance versus time graph shown in figure 4, the Moon's osculating orbit was evident over 500 days. Additionally, when this three-body simulation with 20 digits of floating-point precision was compared to a similar PEP model, after 6000 days the models had a maximum residual disagreement of only 40 microns. This high level of agreement with PEP suggests that this model is an extremely precise model of the three-body Earth, Moon, Sun system with Newtonian gravity.

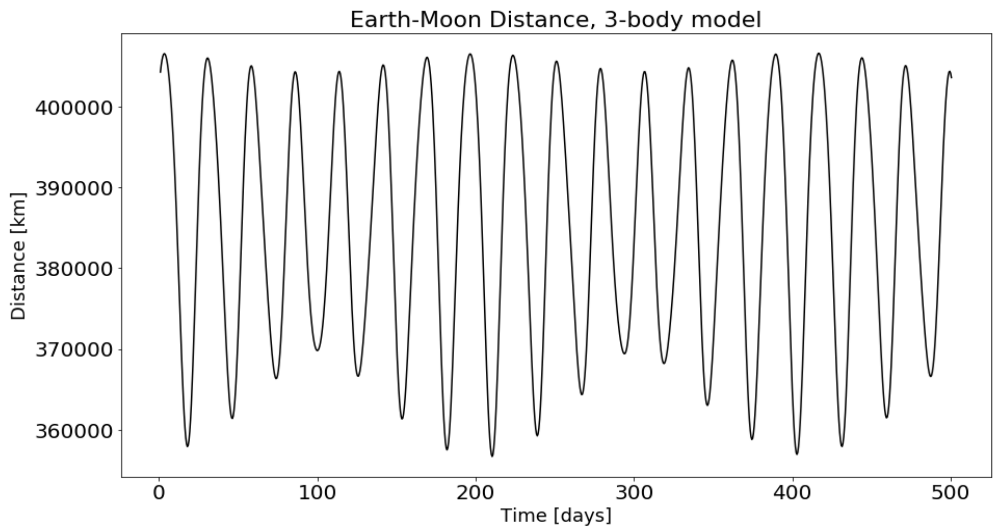

Figure 4: Distance from the Earth to the Moon in kilometers as a function of time in days. The cycles of varying amplitude show the variation (osculation) of the Moon's orbit over time due to the influence of both the Sun and the Moon.

2.1.4. curve fitting

In order to eventually test which theories of gravity best fit LLR data, a least-squared fitting method was implemented in this model.[9] This fitting method will eventually implement the Parametrized Post-Newtonian formalism (PPN) as well as other model assessment strategies to evaluate different theories of gravity.[10] However, since this model currently only uses Newtonian gravity, tests of the least-squared fitting method were performed by comparing simulated data from our model with the model itself and fitting for parameters such as initial positions, initial velocities, and masses. These tests were performed on an even simpler two-body system consisting of the Earth and the Moon. It was discovered that although the method properly fits the model to data, it does not always find the correct minimum of the chi-squared function. The least-squared fitting model works as expected when only one parameter is being evaluated, and it works about 50% of the time when two parameters are being evaluated. For example, a fit for the Earth's x and y velocities resulted in the proper values. However, a fit for Earth's and the Moon's masses resulted in a proposed negative mass of the Moon. Constraints on the parameter values were implemented to try to fix this, but the resulting program could not settle on a minimum. It is possible that the parameters for which the model is fitting are too highly correlated to result in a chi-squared function with an easily-found absolute minimum. In the future, the parameter framework will be changed to strongly reduce the parameter correlations; specifically, Keplerian elements will be implemented for the initial conditions instead of the current Cartesian state vectors. PEP uses these Keplerian elements for its fit, and hopefully these less-dependent parameters will be fit more easily.[11]

While the curve fitting algorithm was being tested and refined, the model of the solar system was also refined so that it could be made more precise and accurate for when the least-squared fitting algorithm could be used more rigorously.

2.2. Adding More Planets

After the simple three-body system was shown to be accurate compared to PEP, more bodies were added to the model and the results were once again compared to similar models within PEP. Firstly, a simulation with five bodies was made. It included the Earth, the Moon, the Sun, Mercury, and Venus. Venus was included because it perturbs the lunar orbit the most after the Earth and the Sun, as indicated in table 1. Mercury was also included because it is easily included in PEP. This model had a similar level of agreement with PEP, showing only a 40-micron discrepancy after a 6000-day integration with `mpmath`.

Jupiter was then added to this simulation because it also perturbs the lunar orbit greatly and because Jupiter is farther away from the Sun than the Earth, meaning that this model would have to deal with larger numbers than before. Specifically, Jupiter is about 5 AU away from the Sun, while the Earth is 1 AU away. Therefore, adding Jupiter to the model means the maximum distances the model must incorporate are five times larger. This indicates that with the same number of precise digits as before, the model may disagree with PEP slightly more. In contrast, when this model was compared to PEP, a similar level of agreement was found; the numbers in the six-body simulation were not large enough to noticeably affect the results. The difference between PEP and this model over 6000 days was only 15 microns. Additionally, when the comparison was tested over 12000 days, a difference of 72 microns was found, as shown in figure 5.

Since this model is accurate with simple five and six body systems, it can be assumed that this model will also be accurate for an arbitrary number of simple bodies. Of course, with the added bodies, the run time of the simulation will increase, and if the added bodies increase the maximum distances in the simulation, a greater number of precise decimal digits may be required to achieve a similar level of accuracy. However, given enough time and enough precise digits, this model can now provide an accurate N-body point particle simulation with Newtonian gravity.

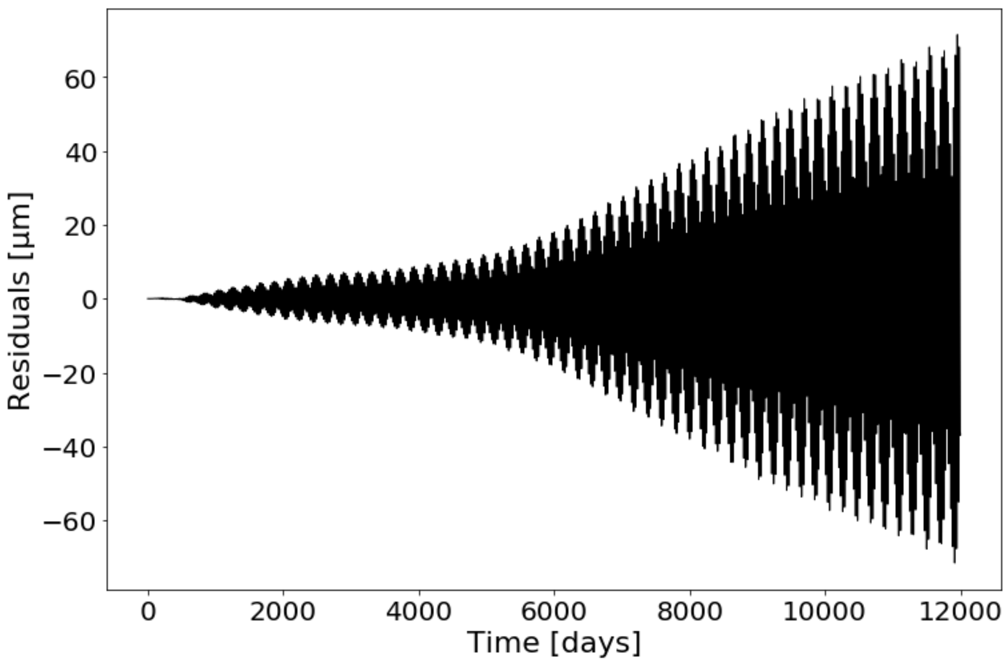

Figure 5: Difference in Earth-Moon distance between this six-body model and PEP over 12,000 days.

2.3. Incorporating An Irregularly Shaped Earth, Moon And Sun

More recently, an irregularly shaped Earth, Moon, and Sun have been added to this model. This was achieved by representing the gravitational potential of the Earth, Moon, and Sun with spherical harmonics, and then by implementing rotation matrices for the Earth and the Moon from the frame defined by the mean equinox and equator to the body frames of the Earth and the Moon, respectively. These rotation matrices were implemented to account for the Earth and the Moon's varying shapes relative to the mean equinox and equator; in contrast, the Sun was assumed to be axially symmetric.

2.3.1. modeling gravitational interactions with spherical harmonics

It can be shown that gravitational potential outside of gravitationally attractive body must always satisfy Laplace's equation.[12] Therefore the gravitational potential can be computationally approximated using a finite series of Legendre polynomials if the body in question has axial symmetry. If the body is not axially symmetric, a series of Legendre polynomials and associated Legendre polynomials must be used. As the Sun was assumed to be axially symmetric, the former process was used to model its gravitational potential, while the latter was used for modelling the Earth and the Moon. PEP's analytic differentiation of gravitational potential to determine acceleration due to a specific body was also used (shown in figure 6), as were PEP's numeric values for the zonal and tesseral harmonic coefficients.[13] Following PEP, the Earth's series approximation included 160 terms, while the Moon's included three terms. The Sun's series included two terms.

$$
\begin{aligned}
\ddot{x}_j = & -\frac{GM_b}{r^3}x_j \\
& + GM_b \sum_{n=2}^{\infty} \left(\frac{a_b}{r}\right)^n \frac{J_n}{r^2} \left[\frac{(n+1)x_j}{r} P_n(\sin\phi) - P_n'(\sin\phi) r \frac{\partial \sin\phi}{\partial x_j} \right] \\
& + GM_b \sum_{n=2}^{\infty} \sum_{h=1}^{n} \left(\frac{a_b}{r}\right)^n \frac{1}{r^2} \Bigg\{ \left[\bar{C}_{nh} \cos h\theta + \bar{S}_{nh} \sin h\theta \right] \left[-\frac{(n+1)x_j}{r} \bar{P}_{nh}(\sin\phi) + \bar{P}_{nh}'(\sin\phi) r \frac{\partial \sin\phi}{\partial x_j} \right] \\
& + h \left[-\bar{C}_{nh} \sin h\theta + \bar{S}_{nh} \cos h\theta \right] \bar{P}(\sin\phi) r \frac{\partial\theta}{\partial x_j} \Bigg\}
\end{aligned}
$$

Figure 6: j acceleration components due to an approximately spherical body b. The corresponding position component of the body being accelerated is given by x_j. The first summation term gives the zonal harmonics. a_b is the radius of the approximately spherical body, P_n is the nth Legendre polynomial, and ϕ is the latitude of the body being accelerated, as viewed from b. J_n is the nth coefficient. The second summation term gives the tesseral harmonics. P_{nh} is the nth, hth associated Legendre polynomial, with cosine coefficient C_{nh} and sine coefficient S_{nh}. This equation uses the normalized version of these. θ is the longitude of the body being accelerated.

2.3.2. implementing rotation matrices for the Earth and the Moon

This model also followed PEP's analytic process for modelling the coordinate rotations of the Earth and the Moon. The body frame of the Earth had its origin at the Earth's center of mass, its vertical axis directed toward the Earth's axis of rotation, and its two other axes lying orthogonal to its vertical axis in typical Cartesian fashion.[14] The rotation into this frame incorporated the Earth's daily rotation and the nutation, precession, and wobble of its axis of rotation. Unlike in PEP, the astronomical parameters required for the calculation of the rotation, nutation, and precession were approximated using series about the epoch J2000. These approximations were found in Astronomical Algorithms.[15] The values required for modelling the Earth's wobble were taken from PEP; however, the only data found was for the years 1956 through 1971. In the future, more contemporary data will be found.

The Moon's body frame was defined similarly to that of the Earth.[16] Its origin was at the Moon's center of mass, and its vertical axis was parallel to the Moon's axis of rotation. The rotation into this frame accounted for the Moon's rotation, precession, and physical librations, as demonstrated in PEP. The astronomical parameters were approximated the same way as for the Earth, and the Moon's physical librations were also determined from Astronomical Algorithms.[17]

2.3.3. results

This version of the model has not yet been compared to PEP. However, a 1300-day simulation of this model was prepared and compared to the version of the model without any irregularly shaped bodies. The difference between the Earth-Moon distance of these models was calculated over time, as shown in Figure 7. The maximum difference was 14 kilometers, or 0.004%.

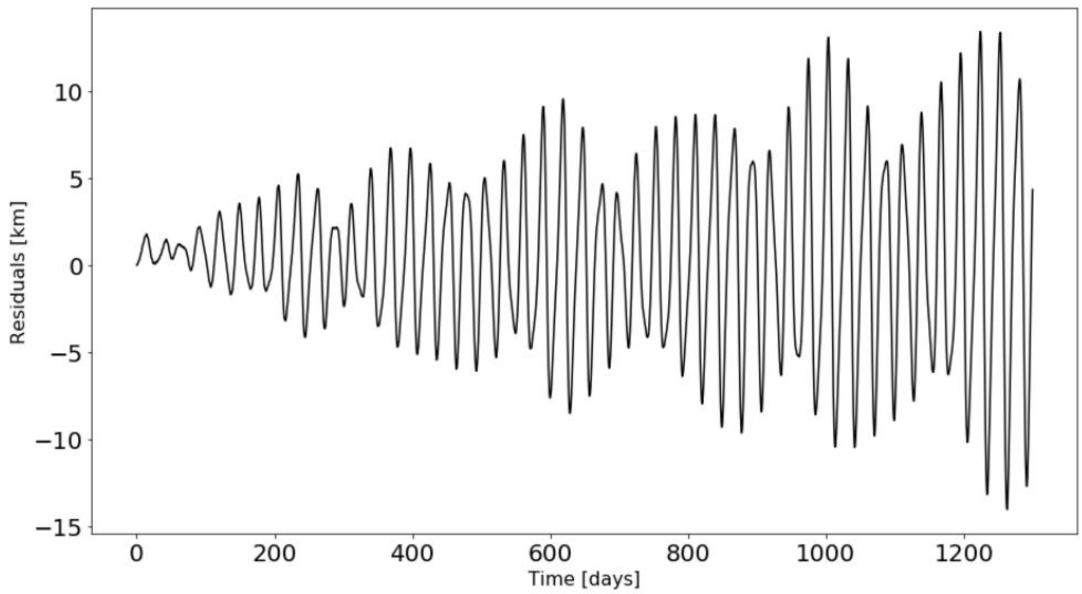

Figure 7: Difference of the Earth-Moon distance between a simulation with an irregularly shaped Earth, Moon, and Sun and a simulation with all bodies perfectly spherically shaped.

3. Conclusion

Currently the computational model consists of 6 bodies; i.e., Earth, Moon, Sun, Mercury, Venus and Jupiter. It accounts for the effects of the irregularities of the Earth, Moon and Sun in shape. It agrees with PEP to 40 micrometers for simple models and to 72 micrometers for a perfectly spherical 6 body system.

The model would be improved by accounting for the locations of the observatory on Earth and the retroreflectors on the Moon in the future. Moreover, it will incorporate different theories of gravitation to look for agreements to experimental data.

4. Acknowledgements

We would like to thank Professor James Battat for his guidance, provision, and support throughout this project. We would also like to thank Dr. John Chandler at the Harvard-Smithsonian Center for Astrophysics for providing us the PEP data at different steps of the model.

Additionally, we would like to thank Wellesley College, Massachusetts Space Grant and MasterCard Foundation for funding our research and supporting our attendance at NCUR. This work was also supported by the National Science Foundation (PHY-1404491) and NASA (NNX-15AC51G).

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