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Verifying Hamiltonicity of Maximum Uniquely Hamiltonian Graphs Quickly

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Abstract

A Hamilton cycle in a graph is a closed walk around the vertices of the graph in which each vertex is visited exactly once. The problem of determining whether an arbitrary input graph contains a Hamilton cycle has been studied extensively and belongs to the so-called NP-complete family of problems. Due to the difficulty of solving this problem for general graphs, we set our sights on a family of graphs described in [J. Sheehan, "Graphs with exactly one Hamiltonian circuit", *J. Graph Theory*, vol. 1 (1977), 37-43]. A 'maximum uniquely Hamiltonian graph' (MUHG) contains the greatest number of edges possible while still admitting a single Hamilton cycle. We present an algorithm that will find the cycle in any of these graphs in polynomial time. Additionally, a proof is presented for the maximum edge case for graphs with exactly two Hamilton cycles. This paper demonstrates that these graphs do not suffer the same complexity issues as do general graphs for the Hamilton cycle problem.

Keywords – Hamilton cycle; cycle exchange; maximum uniquely Hamiltonian graphs

1. Introduction

During the year 1736, Leonhard Euler (see, e.g.,[2]) created a famous puzzle problem involving the bridges of Königsberg. Beginning in some part of Königsberg, could he cross each bridge in town exactly once and return to his starting place? As Euler showed, the Königsberg bridge problem is an ideal situation to model as a graph. The parts of Königsberg are a network of related objects, the relation being the bridges between them. These parts can be represented as vertices (parts of town) and edges (bridges). From this, Euler produced the first theorem of graph theory. The Euler Circuit Theorem states that if, and only if, the number of edges incident on each vertex (the 'degree' of that vertex) is even, there exists an Euler circuit. Now to be clear, the complexity of deciding the existence of an Euler circuit does not coincide with the complexity of solving the Hamilton cycle problem (the former is solvable in linear time, whereas the latter seemingly requires exponential time). However, the Euler circuit problem is analogous to the Hamilton cycle problem and introduces the important idea that examining the vertex degrees is helpful in gaining insight into properties of the graph. This fact will help us greatly when devising an algorithm to find the cycles in our maximum uniquely Hamiltonian graphs.

To reiterate, the Hamilton cycle problem involves walking along edges in the graph, touching each vertex exactly once, and returning to the starting vertex. Modern, non-heuristic algorithms for determining the existence of a Hamilton cycle run in exponential time (implying that even for graphs with relatively few vertices, the problem can take a long time to solve). These algorithms generally employ a method of trying every possible path until a cycle is found, similar to trying every possible combination on a combo lock until that lock opens. One can begin to see why this problem is so challenging. To avoid the challenge of finding such cycles in arbitrary graphs, let's set our focus on a particular family of graphs. The graphs we want to analyze are graphs with exactly one Hamilton cycle. This is a natural starting point in approaching this problem, as one may gain insight into the nature of cycles in graphs.

Currently, solutions to the Hamilton cycle problem prove to be computationally infeasible for arbitrary graphs. If a feasible solution remains to be found, it may be beneficial to focus on graphs with few Hamilton cycles. This paper focuses on a family of graphs containing exactly one Hamilton cycle, however it does not relate to all possible graphs with a single Hamilton cycle. The goal of this paper is to act as a stepping stone in the pursuit of a more feasible solution to the Hamilton cycle problem.

2. Solving the problem for maximum uniquely Hamiltonian graphs

Maximum uniquely Hamiltonian graphs (MUHG's) contain the maximum number of edges while still admitting a single Hamilton cycle. Sheehan [3] discovered that for such a graph with n nodes (where $n \geq 4$), the maximum number of edges it can contain is $[n^2/4] + 1$ (the brackets denoting 'integer part'). Just a few years later, Entringer and Barefoot [1] discovered the size of this family of graphs. The number of MUHG's is a function of the number n of vertices (with $n > 6$), namely $2^{[n/2] - 4}$.

Unlike for arbitrary graphs, the configuration of MUHG's allows for a polynomial computational solving time for answering the Hamilton cycle question. Here below is our algorithm in pseudocode; its initials match those of the author.

2.1 *AAAlgorithm (AAAlg for short)*

```
    while (any_deg > 2) {
  find(maxDegreeNode v);
  if (v isAdjacentTo 2 Deg2Nodes) {
     delete(all_other_edges incident with v);
     update(degree_values);
  } end if;
 } end loop;
```

A written explanation of the code is as follows: while there exists any vertex of degree greater than 2, proceed. Find the maximum degree node in the graph. Ask that vertex if it is adjacent to two vertices of degree 2. If so, proceed in deleting all other edges incident on the high degree vertex, save the edges connecting said vertex to the two degree 2 vertices. We then update the degree values and go back up to the top of the loop, asking again if any vertices of degree greater than 2 exist, and following through as we did before. If, however, all vertices are of degree 2, halt. The algorithm has isolated the Hamilton cycle. See Fig. 1 for an illustration of this process.

Examining the code, let's investigate the approximate run time of this algorithm. The while loop will loop as many times as it takes to obtain a graph where all vertices have degree 2. At most, this will occur n times. Call this approximation O(n) ('big O' n). The find function within the loop simply finds the maximum degree node, an operation running in at most O(n) time. The if statement requires looking at every node adjacent to the maximum degree node. An examination of the degrees of the nodes we are looping over gives us an approximate run time of O(n) per iteration, resulting in a total of about $n^2/2$ (= [n - 1] + [n – 2] + [n – 3] + … + [2]), or $O(n^2)$. This results in an approximate run time of $O(n^2)$, a polynomial upper bound.

How does this algorithm work? Every MUHG has a unique node of highest degree (with value n – 1), called the global maximum. The structure of MUHG's (as proved in [1]) puts this node adjacent to two degree-2 nodes. Note that the edges of any degree-2 node must be used in the cycle we seek. By deleting the other edges incident with our global node (thereby removing edges from our graph that are not used in the cycle), we reduce its degree to 2. After updating the node degrees, we proceed to the next node in line, i.e., the node of second highest degree. This process continues until only the edges necessary for the cycle remain, and we have our Hamilton cycle.

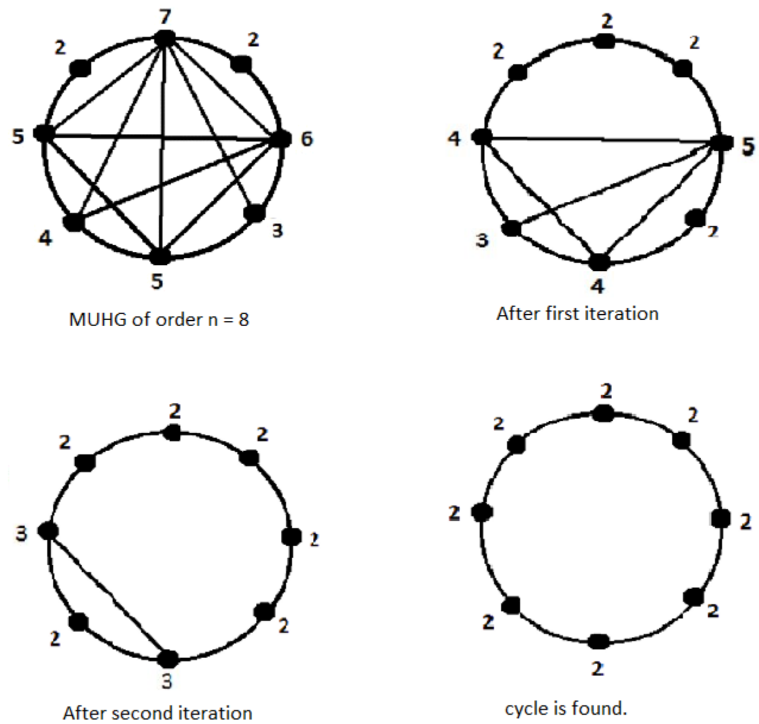

Figure. 1: Running the AAAlgorithm

2.2 Proof of Correctness

We want to show that this algorithm works for every graph in the MUHG family. We proceed using induction. It's important to note that Entringer and Barefoot [1] describe this graph family beginning at two base cases: the even base case graph (of order n = 8) and the odd base case graph (of order n = 7). We have seen our algorithm work on the order n = 8 base case, and similarly, it works on the order n = 7 base case. Now let us assume that it also works on the n – 2 case, and consider the n case for an n > 8. If we can show how to reduce an order-n MUHG to an order n – 2 one, we will have shown that AAAlg works for all cases of MUHG's.

The argument follows the flow of our algorithm. We locate the global maximum degree node (a in Fig. 2) and check for the existence of two adjacent degree-2 nodes. Once this is done, we maintain the edges connecting the global maximum degree node and the degree-2 nodes. We then delete all other edges incident on the max global degree node (b in Fig. 2). Having done this, we contract the edges connecting the previous global max degree node and the degree-2 nodes (c in Fig. 2). This happens to leave us with an MUHG of order n – 2 (d in Fig. 2). By our induction hypothesis, AAAlg works correctly from here to completion, and this demonstrates its correctness for the entire family of graphs. See Fig. 2 for a pictorial presentation of this process.

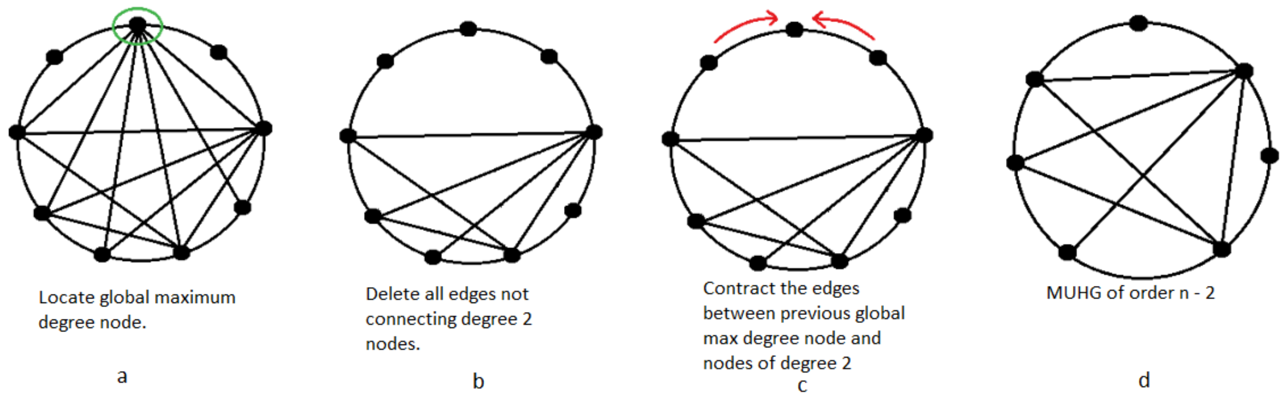

Figure. 2 Reduction from MUHG of order n to order n - 2

3. Edge case for maximum 2-uniquely Hamiltonian graphs

It's natural to wonder if the MUHG can be modified to obtain an extremal graph with exactly two Hamilton cycles, i.e., one with a maximum number of edges subject to this condition. This turns out to be the case. By carefully adding a single edge to an MUHG, we obtain a graph with exactly two Hamilton cycles, and this is the largest number of edges that can be added.

3.1 Proposition

The edge count N + 1 is maximum for graphs with exactly two Hamilton cycles (with $N= [n^2/4] + 1$ again the number of edges in an MUHG).

3.2 Proof

We show that the lower bound N + 1 on edge count is equal to the upper bound. To examine the lower bound, look at Fig. 3 which exemplifies the idea for the order n = 8 MUHG. By adding the edge A, colored green, we achieve the addition of a single cycle. In an MUHG of order n, this second cycle can always be introduced through the addition of an edge between the node of degree n – 2 and the non-adjacent node of degree 2. Through the addition of this single edge, we see that the maximum edge count for a graph with exactly two Hamilton cycles is at least N + 1. Now consider an extremal graph G_2 with exactly two Hamilton cycles named H_1 and H_2 (see Fig. 4). If we remove an edge e from G_2 that is in H_1 but not H_2, we obtain a graph $G_2 - e$ that has exactly one Hamilton cycle (H_2). Thus, $|E(G_2 - e)| \leq N$ by the definition of N, and this shows that $|E(G_2)| \leq N + 1$. This agreement between the lower and upper bounds proves that the maximum number of edges in a graph with exactly two Hamilton cycles is in fact N + 1. QED

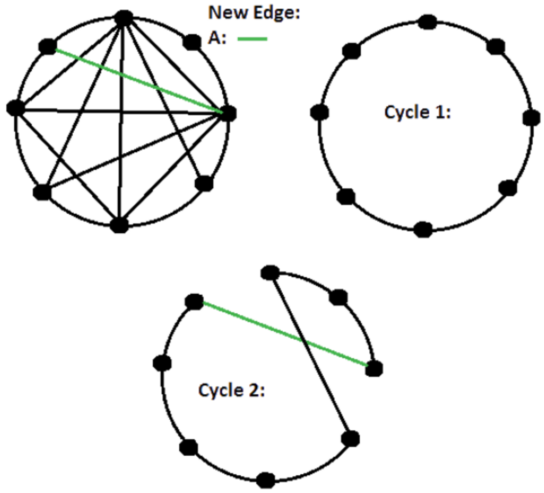

Figure. 3 Addition of edge A introduces new cycle.

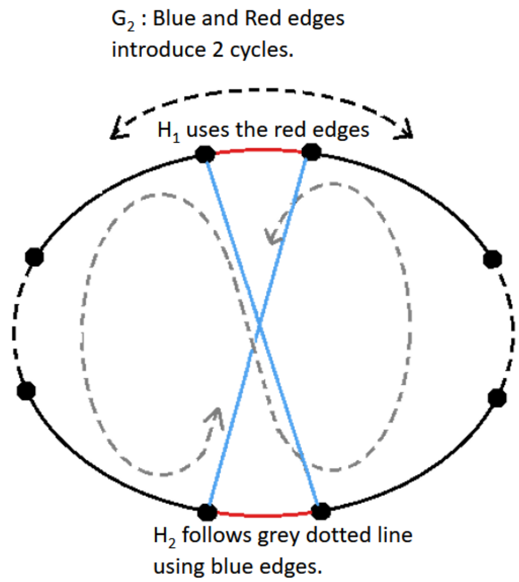

Figure. 4 Graph containing 2 cycles.

4. Discussion

In summary, AAAlg demonstrates that the solution to the Hamilton cycle problem for a MUHG does not belong to the NP-Complete family of problems, and was simply done by counting vertex degrees. This is a significant result, as it shows that there exists a family of graphs that do not suffer the same complexity issues as for arbitrary graphs. In particular, the formation of MUHGs, and any of their subgraphs allow for quick solving times, removing them from the NP-Complete family of problems and into the P family of problems. Further research in this area, which is on-going, may give mathematicians insights into the structure of graphs containing particular numbers of Hamilton cycles. It is the author's personal opinion that determining the number of Hamilton cycles in a graph may be reducible to a cycle exchange counting problem, whereby particular combinations of cycle exchanges allow for different numbers of cycles within a graph. Another significant finding that follows is the discovery of the upper limit for a maximum graph containing exactly two Hamilton cycles, as it can be shown that these graphs too can be solved quickly, because finding a C-4 cycle exchange has proven to be trivial, though that work is saved for a future paper.

We have devised a polynomial-time algorithm to find both cycles in a 2-uniquely Hamiltonian graph and plan to detail it in a future paper.

5. References

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