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Decidability of Cup Problems and Computation of Solutions

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Abstract

An interesting problem in recreational mathematics is the three cups problem. It is a problem without a solution. There are infinitely many, related, n-choose-r cup problems. Each of these is shown to be decidable, through the construction of an explicit algorithm. The algorithm is able to compute solutions for solvable problems. The algorithm is run for $0 \leq r \leq n \leq 10$ and the results are discussed.

Keywords: Puzzles, Algorithms, Solvability

1. Introduction

The 3 cups problem provides an elementary example of a puzzle that can not be solved.

Definition 1.1. The **three cups problem** begins with three cups, each upside down on a table, and it asks for a series of moves, each of which turns over exactly two cups. The objective is to arrive at an arrangement where all three cups are right-side up on the table.

We show in Section 3 that no such series of moves exists. However, our main concern deals with more general problems.

Definition 1.2. For $0 \leq r \leq n < \infty$, an **n-choose-r cups problem** begins with n cups, each upside down on a table, and it asks for a series of moves, each of which turns over exactly r cups. The objective is to arrive at an arrangement where all n cups are right-side up on the table.

Upon restriction of the set of allowable moves, one can produce necessary and sufficient conditions to determine which cup problems are solvable. This theory is developed in Section 4. However, the general decision problem to determine which (n, r) pairs result in solvable cup problems remains complicated. Each problem is decidable, and we demonstrate this fact in Section 5. The algorithm that we develop can also compute solutions, when they exist. An example problem is solved in Section 6. Extensions to the theory are discussed in Section 7.

2. Terminology

We begin with three definitions.

Definition 2.1. A **puzzle** is a question, problem, or contrivance designed for testing ingenuity [5].

We thus treat a given cup problem as a specific type of puzzle—a mathematical problem. Not all such problems admit solutions.

Definition 2.2. A **solution** is an answer to a problem [8].

Definition 2.3. A problem is said to be **solvable** if it is susceptible of solution [9].

To better appreciate how mathematical problems can be unsolvable, consider the historical problem of searching for a formula to express the roots of higher-degree polynomials. While

$$ax^2 + bx + c = 0 \tag{1}$$

can be solved with

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{2}$$

and similar, yet more complicated formulas exist for cubic and quartic equations, the following theorem can be stated.

Theorem 2.4. *There exists no general algebraic solution to polynomial equations of degree five or higher.*

Proof. See Section 7.7 of Fine and Rosenberger [3]. □

Solvability is related to decidability.

Definition 2.5. A **decision problem** is a problem with a yes or no answer [10].

Definition 2.6. A decision problem is **decidable** if an algorithm exists that can and will answer the problem [10].

Cup problems present an infinite set of decision problems. For each (n, r) pair we attempt to decide if the associated cup problem is solvable or not solvable.

3. Three Cups

Theorem 3.1. *The three cups problem is not solvable.*

Proof. Let 0 represent an upside down cup, and 1 represent a right-side-up cup. We have available the following moves:

$$m_1 : (0,0,0) \mapsto (1,1,0)$$
$$m_2 : (0,0,0) \mapsto (1,0,1)$$
$$m_3 : (0,0,0) \mapsto (0,1,1).$$

Note that composition of moves is commutative (see [7]). Thus, irrespective of order, for a series of moves, let x_i denote the number of times that m_i has been employed. Using matrix multiplication (see [4]) and modular arithmetic (see [1]) we then express any solution as $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{N}^3$, where

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \pmod 2. \tag{3}$$

Not all x_i can be congruent to zero, which means without loss of generality that $x_3 \equiv 1 \pmod 2$. Consideration of row three of (3) then implies that $x_2 \equiv 0 \pmod 2$, and additional consideration of row two of (3) then implies $x_1 \equiv 0 \pmod 2$. This results in a failure to satisfy row one of (3).

Thus we conclude that the three cups problem is unsolvable. □

4. Adjacent Cups

As a step between the three cups problem and the more general n-choose-r cups problem, we consider here a simplified n-choose-r cups problem, where we restrict the set of allowable moves.

Definition 4.1. For $0 < r < n < \infty$, a **simplified n-choose-r cups problem** begins with n cups, each upside down on a a table, and asks for a series of moves, each of which turns over exactly r *adjacent* cups. The objective is to arrive at an arrangement where all n cups are right-side up on the table.

Theorem 4.2. *The simplified n-choose-r cups problem is solvable if and only if* $r \mid n$.

Proof. As in the proof of Theorem 3.1 we seek a solution of a matrix equation. The relevant matrix has n rows, $n-r+1$ columns and the following form:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ddots & 1 & 0 & 0 \\ 1 & 1 & 1 & \ddots & 1 & 1 & 0 \\ 1 & 1 & 1 & \ddots & 1 & 1 & 1 \\ 0 & 1 & 1 & \ddots & 1 & 1 & 1 \\ 0 & 0 & 1 & \ddots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-r-1} \\ x_{n-r} \\ x_{n-r+1} \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix} \pmod 2. \tag{4}$$

This represents a determined system of equations and a solution can be read off from the bottom right using back substitution (see [4]). It is evident that for $k \geq 0$ with $k \equiv 0 \pmod r$ that we have $x_{n-1+r-k} = 1$ and all other entries of $\mathbf{x} = (x_1, x_2, ..., x_{n-r}, x_{n-r+1})$ must be 0. Working up and to the left then reveals, upon consideration of x_1, that a solution exists if and only if $r \mid n$. ☐

5. Decidability

For a general n-choose-r cup problem, the following expression determines the number of allowable moves (see Section 5.7 of [2]):

$$N(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}. \tag{5}$$

For large values of n the value for N can be extremely large, and it would be complicated to write a general yet detailed matrix equation analogous to that displayed in (4). We instead use recursion.

5.1. the recursively constructed array

We desire a four-dimensional array X, where each $X[n, r+1, i, j]$ is a number and each $X[n, r+1, \cdot, \cdot]$ is a matrix with columns representing the allowable moves of the n-choose-r cups problem.

We construct such an array, first for $n = 2$ and $r = 0, 1, 2$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We then utilize the following formula (see [3]),

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}, \tag{6}$$

to recursively define $X[n, \cdot, \cdot, \cdot]$ for successively large values of n. The results are (for $n = 3$ and $r = 0, 1, 2, 3$)

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and (for $n = 4$ and $r = 0, 1, 2, 3, 4$)

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&1&0&1\\0&0&1&0&1&1\\1&0&0&1&1&0\\1&1&1&0&0&0\end{bmatrix}, \begin{bmatrix}1&0&1&1\\0&1&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix},$$

and so on.

This construction can be carried out within R (see [6]). The following program proceeds up until $n = 10$.

```
m<-10
s<-choose(m,floor(m/2))
v<-numeric(m*(m+1)*m*s)
X<-array(v,dim=c(m,m+1,m,s))
for (i in 1:2) {for (j in 1:1) {X[2,1,i,j]=0}}
for (i in 1:2) {for (j in 1:2) {X[2,2,i,j]=1-abs(i-j)}}
for (i in 1:2) {for (j in 1:1) {X[2,3,i,j]=1}}
for (n in 3:m) {for (i in 1:n) {X[n,1,i,1]=0}}
for (n in 3:m) {for (i in 1:n) {X[n,n+1,i,1]=1}}
row = function(n,r) c(rep(1,choose(n-1,r-2)),rep(0,choose(n-1,r-1)))
combine = function(a,b,c) rbind(cbind(a,b),c)
for (n in 3:m) {for (r in 2:n)
    {X[n,r,1:n,1:(choose(n,r-1))]=
    combine(X[n-1,r-1,1:(n-1),1:(choose((n-1),r-2))],
    X[n-1,r,1:(n-1),1:(choose((n-1),(r-1)))],row(n,r))}}
```

In principle, we could have set m equal to any positive integer.

5.2. the decision procedure

We next decide which matrices within the array represent cup problems that are solvable. The following lemma is useful.

Lemma 5.1. *Let A be an $m \times n$ matrix, and let A_e denote its echelon form. For the augmented matrix $[A|\mathbf{b}]$, denote its echelon form with $[A_e|\mathbf{b_e}]$. Then $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{N}^n$ solves*

$$A\mathbf{x} \equiv \mathbf{b} \pmod{\mathbf{2}} \tag{7}$$

if and only if it solves

$$A_e\mathbf{x} \equiv \mathbf{b_e} \pmod{\mathbf{2}}. \tag{8}$$

Proof. The echelon form of a matrix is obtained through elementary row operations [4]. These operations consist of

(i) A row within the matrix can be switched with another row

(ii) Each element in a row can be multiplied by a nonzero constant

(iii) A row can be replaced by the sum of that row and a multiple of another row.

We apply these row operations to an augmented matrix and speak of the solution space of its associated matrix equation.

(i) clearly leaves the solution space unchanged. Since we are working mod 2, the only nonzero constant is "1" and thus (ii) leaves the solution space unchanged as well. Finally, (iii) can be seen to leave the solution space unchanged by noting that the only nontrivial option (mod 2) involves the replacement of a row with the sum of that row and another row. This can be expressed symbolically as

$$R_i + R_j \to R_i.$$

Since $(R_i + R_j) + R_j = R_i + (R_j + R_j) \equiv R_i \pmod 2$ we see that the operation is invertable and **x** satisfies both R_i and R_j if and only if **x** satisfies $R_i + R_j$ and R_j.

Thus, in all three cases the solution space is left unchanged, and it thus remains unchanged even after any (finite) series of row operations, including those used to compute $[A_e|\mathbf{b_e}]$ from $[A|\mathbf{b}]$. ☐

Our decision procedure can be carried out using R. The following program operates on an augmented matrix, reduces it to its row echelon form, and then decides whether it represents a solvable system of equations or not.

```
swap = function(M,i,j,verbose=F) {M[c(i,j),]=M[c(j,i),];return(M)}
solvable = function(M) {
   m=dim(M)[1]; n=dim(M)[2]; s=min(dim(M)); i=1; j=1
      while (i<m & j<n) {
         if (M[i,j]==0) {
            if (sum(M[(i:m),j])==0) j=j+1
            else {for (s in 1:1) {while (M[(i+s),j]==0) s=s+1};M=swap(M,i,i+s);}}
         else {for (t in 1:(m-i)) {for (v in 0:(n-j))
         {M[i+t,j:n]=abs(M[i+t,j:n]-M[i+t,j]*M[i,j:n])}};i=i+1;j=j+1}}
   if (j<n & M[m,n]==0) print(1)
   else {if (j<n & M[m,n]==1) print(max(M[m,1:(n-1)]))
   else print(1-max(M[i:m,n]))}}
```

The program can be used to define a function that operates on matrices within the array X.

```
solvablex = function(k,l)
solvable(cbind(X[k,l,1:k,1:(choose(k,l-1))],rep(1,k)))
```

After applying "solvablex" for various values of $k = n$ and $l = r + 1$ we obtain data with regards to solvability. These data are displayed in Table 1.

Also note that in principle for any (n, r) such that $0 \leq r \leq n < \infty$ we can construct $X[n, r + 1, \cdot, \cdot]$ and then compute solvablex$(n, r + 1)$. The nature of this procedure is that it will always terminate, and we thus have an algorithm establishing decidability.

Theorem 5.2. *For $0 \leq r \leq n < \infty$ the n-choose-r cups problem is decidable.*

6. Solutions

The algorithms used in Section 5 not only demonstrate decidability of cup problems, they also provide us with a decision procedure. Furthermore, this procedure can be modified so as to produce explicit solutions to solvable cup problems. We demonstrate with an example.

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$
$n=1$	0	1	-	-	-	-	-	-	-	-	-
$n=2$	0	1	1	-	-	-	-	-	-	-	-
$n=3$	0	1	0	1	-	-	-	-	-	-	-
$n=4$	0	1	1	1	1	-	-	-	-	-	-
$n=5$	0	1	0	1	0	1	-	-	-	-	-
$n=6$	0	1	1	1	1	1	1	-	-	-	-
$n=7$	0	1	0	1	0	1	0	1	-	-	-
$n=8$	0	1	1	1	1	1	1	1	1	-	-
$n=9$	0	1	0	1	0	1	0	1	0	1	-
$n=10$	0	1	1	1	1	1	1	1	1	1	1

Table 1: Results of our decision procedure for various n-choose-r cups problems: "1" indicates solvable, "0" indicates not solvable, "-" indicates not applicable

Table 1 shows that the 5-choose-3 problem is solvable. The associated matrix equation is

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \pmod 2 \tag{9}$$

and the augmented matrix is

$$M = \left[\begin{array}{cccccccccc|c} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{10}$$

In reduced form it becomes

$$M_e = \left[\begin{array}{cccccccccc|c} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right], \tag{11}$$

which represents the following simpler (and equivalent) system of equations:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \pmod 2. \tag{12}$$

Now solvability is evident and we can read off a simple solution, namely

$$(x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0, x_9 = 0, x_{10} = 0).$$

Lemma 9 in reverse assures that employing the moves represented by the first three columns of the matrix in (9) results in a solution. This can be verified using five fingers on one hand. Try it.

7. Extensions

We conclude this paper with a concrete question and an abstract challenge. The question involves our collected data.

- Does the apparent pattern on display in Table 1 continue on for larger values of n and r?

  Hint: Consider simple sequences of moves as whole composite moves.

The challenge involves generalizations of cup problems. Instead of using cups, which have two states (up and down), we can use triangles, or arbitrary, regular, convex polygons, or even more abstract mathematical objects. How complicated must the mathematics become before we can accomplish the following?

- Construct an undecidable puzzle.

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