Proceedings of The National Conference
On Undergraduate Research (NCUR) 2013
University of Wisconsin, La-Crosse, WI
April 11-13, 2013

Structures of Interest for Interval 3-graphs

Joshua Reinoehl
Mathematics
University of North Alabama
One Harrison Plaza
Florence Alabama, 35632

Faculty Advisor: Dr. Shilpa Dasgupta

Abstract

A bipartite graph $G(X, Y, E)$ is an interval bi-graph if to every vertex, $v \in V(G)$, we can assign an interval on the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y = \phi$ and $x \in X$ and $y \in Y$. There is a significant amount of research currently being conducted in the area of interval bi-graphs. In this paper we will look at interval 3-graphs. These graphs are obtained by adding an additional partite set to an interval bi- graph. Here we find a forbidden sub-graph of interval 3-graphs as well as some properties of a special classification of an interval 3-graph.

Keywords: Structures, Interval bi-graphs, Bipartite

1. Introduction

Interval graphs were introduced by Hajos [2] and were then characterized by the absence of induced cycles of length larger than 3 and asteroidal triples by Lekkerkerker and Boland [6] in 1962. These graphs are used to provide numerous models in diverse areas such as genetics, psychology, sociology, archaeology, or scheduling. Other useful characterizations of interval graphs were given by Gilmore and Hofman in 1964 [13] and Fulkerson and Gross [12]. For more details on interval graphs and their applications, see books by Roberts [16], Golumbic [14] and Mckee and McMorris [15]. So far no complete forbidden sub-graph characterizations of interval bi-graphs has been found, but initially it was thought that asteroidal triples of edges along with induced cycles larger than 4 would work. They proved that if B is an interval bi-graph then B does not have *asteroidal triple of edges*(ATE). An ATE is a set of three edges such that for any two there is a path from the vertex set of one to the vertex set of the other that avoids the neighborhood of the third edge. However, Muller [7] found insects and Hell and Huang [4] found edge asteroids and bugs as forbidden sub-graphs, and to date a complete characterization is still not available.

2. Preliminaries

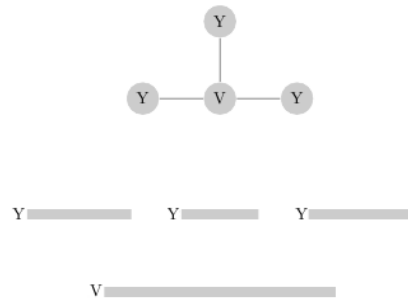

Figure 1: The interval representation of a complete $K_{1,3}$ graph

2.1 Definition 1.

Let a graph G have vertex set V and edge set E. If $x, y \in V$ are adjacent, then we denote $xy \in E$. A finite simple graph G(V,E) is an interval graph if we can find a mapping $\theta: v \rightarrow I_v$ from vertices of G to intervals on the real line such that the edge xy exists if and only if $I_x \cap I_y = \phi$ for all $x, y \in V(G)$.

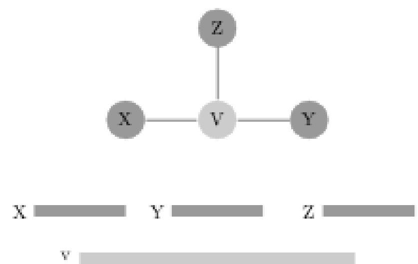

Figure 2: The interval representation of a complete bipartite $K_{1,3}$ graph

2.2 Definition 2.

A bipartite graph $G(X, Y, E)$ is an interval bi-graph if to every vertex, $v \in V(G)$, we can assign an interval on the real line, I_v, such that $xy \in E(G)$ if and only if $I_x \cap I_y = \phi$ and $x \in X$ and $y \in Y$.

2.3 Definition 3.

A Star with n+1 vertices is a complete $k_{1,n}$ as shown in Figure 3.

2.4 Definition 4.

A *ClawGraph* is a Star S_n where $n = 3$ as seen in Figure 3.

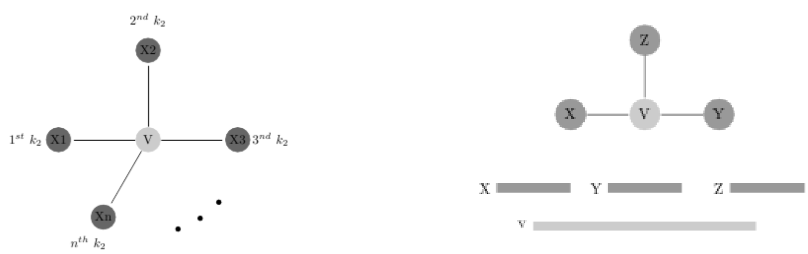

Figure 3: $S3 \equiv Claw$

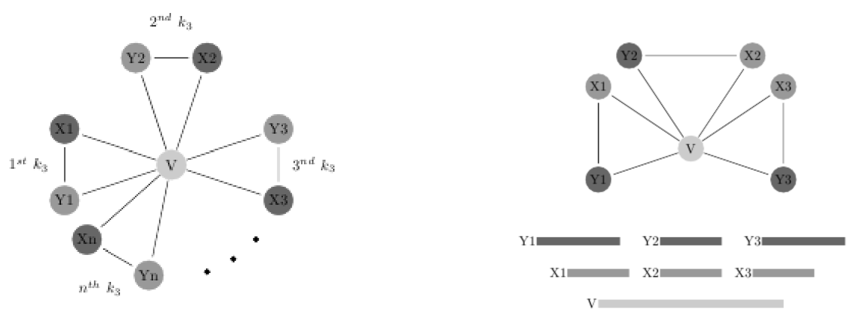

Figure 4: 2-$Star3 \equiv$ 2-$Claw$

2.5 Definition 5.

A 2-$Star_n$ is a graph with n k_3s which share a common vertex, as shown in Figure 4.

2.6 Definition 6.

A 2-*ClawGraph* is a 2-$star_n$ where $n = 3$, as seen in Figure 4.

Now we will add vertices to the different components of the central vertex of a 2-*ClawGraph* to obtain our graphs of interest. Each time a new set of vertices is added they shall be called the terminating vertices for the particular structure they create. The newly formed edges shall be known as the terminating edges. Also, the newly formed k3s shall be known as the terminating k3s.

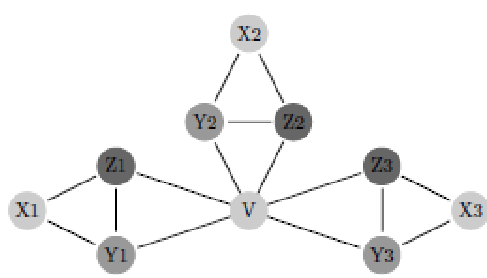

Figure 5: 1-*Extended* 2-*ClawGraph*

2.7 Definition 7.

A 1-*Extended* 2-*ClawGraph* is a graph in which 3 vertices (x_i; i=1,2,3) are made adjacent to the vertices of the edges of the 3 k_3s of a 2-*ClawGraph* such that the x_is $\Leftrightarrow$ v for $i = 1, 2, 3$ where v is the central vertex (the k_3s that this formed are the terminating k_3s of the graph).

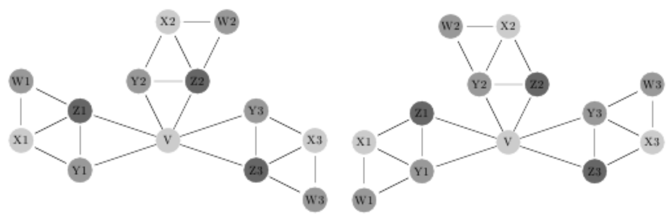

Figure 6: 2-*Extended* 2-*ClawGraph*

2.8 Definition 8.

A 2-*Extended* 2-*ClawGraph* is a graph in which 3 vertices (w_i; i=1,2,3) are made adjacent to the vertices of the edges of the 3 terminating k_3s of a 1-*Extended* 2-*ClawGraph* as seen in Figure 6. It can be noted here as seen in Figure 6 that w_i; $i = 1, 2, 3$ are the new terminating vertices, and since w_i can be adjacent to either x_i, y_i where $i = 1, 2, 3$ or x_i, z_i where $i = 1, 2, 3$ and so either x_i, y_i, w_i where $i = 1, 2, 3$ or x_i, z_i, w_i where $i = 1, 2, 3$ becomes the new terminating k_3s.

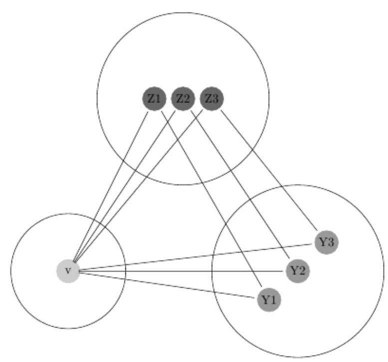

Figure 7: tripartite representation of a 2-*ClawGraph*

2.9 Definition 9.

A tripartite graph $T\,(X, Y, Z, E)$ is an interval 3-graph if to every vertex, $v \in V(T)$, we can assign an interval of the real line, I_v, such that $x, y \in E(T)$ and $y, z \in E(T)$ and $x, z \in E(T)$ if and only if $I_x \cap I_y = \emptyset$ and $I_y \cap I_z = \emptyset$ and $I_x \cap I_z = \emptyset$ and $x \in X$ and $y \in Y$ and $z \in Z$ A tripartite graph $T\,(X, Y, Z, E)$ is an interval 3-graph if to every vertex, $v \in V(T)$, we can assign an interval of the real line, I_v, such that $x, y \in E(T)$ and $y, z \in E(T)$ and $x, z \in E(T)$ if and only if $I_x \cap I_y = \emptyset$ and $I_y \cap I_z = \emptyset$ and $I_x \cap I_z = \emptyset$ and $x \in X$ and $y \in Y$ and $z \in Z$.

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3. Interval 3-graphs

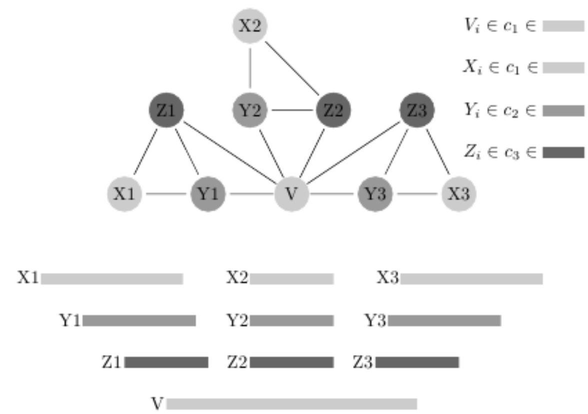

Figure 8: A representation of a 1-*Extended* 2-*ClawGraph*, known as T_*

3.1 Lemma 1.

The 1-*Extended* 2-*ClawGraph* which we will denote by T_*, as seen in Figure 8, is an Interval 3–graph.

Proof. We will prove this by giving an interval 3-graph representation of the 1-*Extended* 2-*ClawGraph*. Let the vertex $v \in c_1$. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, and $Z = \{z_1, z_2, z_3\}$. Next let $Y \in c_2$, note that $c_1 = c_2$ because $v \Leftrightarrow$ the vertices in $Y \Rightarrow I_v \cap I_{y_i} = \phi$ where $i = 1, 2, 3$ and the color class of Y = the color class of v. Next let $Z \in c_3$, note that $c_1 = c_2 = c_3$ because $z_1 \Leftrightarrow y_1$, $z_2 \Leftrightarrow y_2$, $z_3 \Leftrightarrow y_3 \Rightarrow I_{z_i} \cap I_{y_i} = \phi$ where $i = 1, 2, 3$ and the vertices of Y and the vertices of Z belong to a different color class, and the vertices of $Z \Leftrightarrow v \Rightarrow I_{z_i} \cap I_v = \phi$ where $i = 1, 2, 3$ and the vertices of Z and the vertex v belong to a different color class. Next we color X, Note that the vertices in $X \Leftrightarrow v \Rightarrow I_{x_i} \cap I_v = \phi$ where $i = 1, 2, 3$ or the color class of X is the same as the color class of v but, $I_{x_i} \cap I_v = \phi$ where $i = 1, 2, 3$ is forced for at least one x_i, $i = 1, 2, 3$ but $x_i \Leftrightarrow v \Leftarrow$ that x_i must share the same color class as v thus $X \in c_1$.

3.2 Lemma 2.

In an Interval 3-graph representation of a 1-*Extended* 2-*Claw Graph*, as shown in Figure 8, the terminating vertices will always share the same color class as the central vertex.

Proof. Let us consider the 1-*Extended* 2-*Claw Graph* T_*, given in Figure 8, whose terminating vertices are x_1, x_2, x_3. Let the central vertex v belong to the color class c_1. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, and $Z = \{z_1, z_2, z_3\}$. Let the vertices of Y belong to the color class c_2 and let the vertices of Z belong to the color class c_3. Since the vertices of X are adjacent to the vertices of Y, the color class of $X = c_2$ = the color class of Y. Since the vertices of X are adjacent to the vertices of Z, the color class of $X = c_3$ = the color class of Z. $\therefore$ the vertices of X belong to the color class c_1, which is the same as the color class of the central vertex v. Thus, we do the interval 3-graph representation of T_* such that vertices v, x_1, x_2, x_3 belong to the same color class c_1. Even though $I_{x_2} \cap I_v = \phi$ is forced, as shown in Figure 8, it does not imply adjacency between x_2 and v since they belong to the same color class. Therefore, the terminating vertices x_1, x_2, x_3 of 1- *Extended* 2-*Claw Graph* T_* will always share the same color class as the central vertex v.

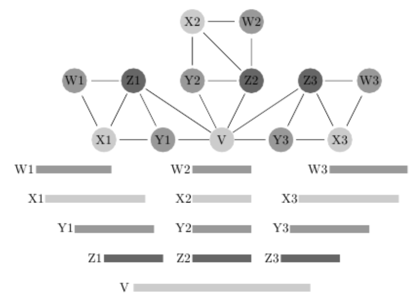

Figure 9: A representation of a the 2-*Extended* 2-*ClawGraph*, known as $T1$

3.3 Theorem 1. An interval 3-Graph cannot have T1, as shown in Figure 9, as an induced subgraph.

Proof. We will use Figure 8 and Figure 9 to prove Theorem 1. Assume there exists an interval graph G such that T_1 as given in Figure 9 is an induced subgraph of G. Thus T_1 must have an interval 3-graph representation. It can be seen by comparing Figure 8 and Figure 9, that T_1 can be constructed by adding three new vertices w_1, w_2, w_3 to $T*$ (from Figure 8) such that each component of v gets a new vertex as shown in Figure 9. Let $W = \{w_1, w_2, w_3\}$. We now look at the interval 3-graph representation of T_1. To do this we take the representation of T_* from Figure 8 and add three intervals corresponding to vertices w_i where $i = 1, 2, 3$ as shown in Figure 9. We now choose the color class of the vertices of W. Since $w_i \Leftrightarrow x_i$, $w_i \Leftrightarrow z_i$; $i = 1, 2, 3$ so the color class of W must not be c_1(which is the color class of the vertices of X) or c_3(which is the color class of the vertices of Z) and hence it should be c_2. Note that the vertices of W are not adjacent to v so I_{w_1} is drawn such that $r(I_{w_1}) < l(I_v)$ and I_{w_3} is drawn such that $l(I_{w_3}) > r(I_v)$ as shown in Figure 9. Now we look at the vertex w_2. We know that the color class of w_i; $i = 1, 2, 3$ is the same as the color class of y_i; $i = 1, 2, 3$ which is c_2. It can be easily seen from Figure 9 that $I_{w_2} \cap I_v = \phi$ is forced. We also know that the color class of $w_2 = c_2 = c_1 =$ the color class of v. This implies that w_2 must be adjacent to v which is a contradiction. Hence T_1 is not an interval 3-graph.

4. Conclusion

In future work, we will examine different properties of interval 3-graphs and extend the current results to interval k-graphs. Interval k-graphs are graphs with proper coloring where each vertex v can be assigned an interval I_v of the real line such that two vertices are adjacent if and only if their corresponding intervals overlap and each vertex has a different color.

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