# Isomorphic Circulant Graphs and Applications to Homogeneous Linear Systems 

Shealyn Tucker<br>The Department of Mathematics and Computer Science<br>Salisbury University<br>1011 Camden Ave.<br>Salisbury, Maryland 21801 USA<br>Faculty Advisor: Dr. Michael Bardzell


#### Abstract

In this research project we will investigate patterns in numerical data regarding the number of isomorphic circulant graphs for a given number of vertices. There is no known formula for this, but computer computations have provided useful data which has not been fully analyzed. These graphs will also be used to model certain circulant linear systems of equations to describe their decomposition into subsystems and, ultimately, provide information about the nature of their solution sets. In order to obtain results, detailed illustrations of the isomorphic circulant graphs will be carefully analyzed as well as their corresponding circulant linear systems of equations. The software system Mathematica will be used to generate larger circulant graphs and certain functions of the program will determine the behaviors of more complicated circulant linear systems of equations. The Online Encyclopedia of Integer Sequences will be accessed for information relevant to pattern recognition in numerical isomorphism class data. Four lemmas and a theorem will be presented pertaining to the behavior of the isomorphic circulant graphs, where each vertex within the graph is connected to one other vertex within the graph. The behaviors of the circulant graphs vary based on the number of vertices or the order $m$ of the graph, the distance in between each vertex, whether or not the graph breaks into cycles, and how many edges are connecting the vertices within the cycles. We anticipate these theorems will help us generate number theoretic formulas for isomorphic circulant graphs, where each vertex within the graph is connected to two or more vertices within the same graph.


## Keywords: Homogeneous, Isomorphic, Circulant

## 1. Introduction

Graph theory is useful for a plethora of applications where the considerations are a finite collection of objects and their relative connections to each other. For example, Dijkstra's algorithm is used to find the shortest path in a graph ${ }^{5}$. This technique is used to implement a service like MapQuest which finds the shortest route to drive between two points on the map. Dijkstra's algorithm can also be used to solve problems, such as, network routing; where the goal is to find the shortest path for data packets to take through a switching network. Flow or transportation networks can be modeled with directed weighted graphs (the edges have orientations and numerical weights), where the weights depict the capacity that an edge can hold and the orientation of the edge shows which direction the substance is flowing ${ }^{6}$. Such methods are used to direct the flow of fluids in pipes and electric circuits, map the path of flights between the airports of the world, model the traffic flow on a new highway or road system, and even track cyber terrorists and high risk computer hackers. Graph "coloring" systems are used in scheduling complicated events such as filling in time slots at a corporation. Grocery store chains even use graph theory to determine optimal product placement based on consumer purchasing behavior, as shown in Figure 1.


Figure 1. A graph depicting how products are placed based on what the customers buy at the grocery store.
Figure 1. Graphs are used to describe the most profitable product placement strategy. An edge between the vertices reveals that the paired items are more likely to be bought together. For example, a person who purchases bacon is more likely to also buy mushrooms than they are an avocado. By correlating vast amounts of data recorded over time and organizing that data into a large (much larger than the one above) graph, sophisticated algorithms can be implemented on a computer to determine optimal food placements within a store.

Graph theory has many applications, but there are still some applications that have yet to be discovered. The core of this research involves isomorphic circulant graphs and how the graphs can be applied to homogeneous systems of linear equations and vice versa; in order to produce a formula that describes the number of isomorphic graphs with a given number of vertices. In sections 1.1 through 1.3, a detailed description of isomorphic graphs, circulant graphs, and isomorphic circulant graphs will be provided. The circulant systems of linear equations will be discussed in section 1.4.

### 1.1 Isomorphic Graphs

In discrete mathematics a graph, $\mathrm{G}=(V, E, \phi)$ consists of a finite nonempty set $V$ of vertices, a finite set $E$ of edges, and an incidence function $\phi$ that maps edges to unordered pairs of vertices. Two vertices, $v$ and $w$, are adjacent if there is an edge, $e \in E$ for which, $\phi(e)=v w$. This makes the vertices incident with the edge. In simpler terms, a graph is a diagram that contains a collection of vertices connected by line segments or arcs known as edges ${ }^{3}$. Two graphs, $\mathrm{G}=(V, E, \phi)$ and $\mathrm{G}^{\prime}=\left(V^{\prime}, E^{\prime}, \phi^{\prime}\right)$ are isomorphic if there exists a one-to-one and onto function $\zeta: \mathrm{V} \rightarrow \mathrm{V}^{\prime}$ such that $v_{i}$ and $v_{j}$ are adjacent in $G$ if and only if $\zeta\left(v_{i}\right)$ and $\zeta\left(v_{j}\right)$ are adjacent in $\mathrm{G}^{\prime 1}$. This means that isomorphic graphs can be re-labeled in such a way that they become equivalent graphs. Isomorphic graphs do not always visually appear the same, but if they have the same number of vertices, edges, and identical adjacency relationships, then they are the same ${ }^{1,3}$, as shown in Figure 2.


Figure 2. The image is of three isomorphic circulant graphs.
Figure 2. Each graph has four vertices, four edges, and identical adjacency relationships. Therefore, by the definition of isomorphic graphs, these three graphs are isomorphic even though the graphs do not visually look similar.

### 1.2 Circulant Graphs

For this research, certain highly symmetric graphs will be studied. A circulant graph is a graph where every vertex is connected to the same set of relative vertices. We begin with $m$ vertices and arrange them in a cycle. Now suppose each vertex is connected via edges to the vertices $i_{1}, i_{2}, \ldots, i_{r}$ positions around the cycle reading counterclockwise, where $i_{l}, i_{2}, \ldots, i_{r}$ are positive integers. This uniquely defines a circulant graph denoted $\mathrm{C}\left[n,\left\{i_{1}, i_{2}, \ldots, i_{r}\right\}\right]$. Here $\left\{i_{1}, i_{2}\right.$, $\left.\ldots, i_{r}\right\}$ are called the keys to the circulant graph. The keys provide complete information on all vertex adjacencies ${ }^{2}$. For example, $\mathrm{C}[7,\{1,3\}]$ is the circulant graph with seven vertices, where each vertex is connected to two other vertices, one and three positions over, respectively.


Figure 3: The circulant graph $\mathrm{C}[7,\{1,3\}]$.

### 1.3 Isomorphic Circulant Graphs

A problem posed by Dr. Paul Kehle, who is an assistant professor at the Hobart and William Smith Colleges, is to find a formula which produces the number of non-isomorphic circulant graphs as a function of the number of vertices ${ }^{2}$. Such a function has yet to be discovered. Computer computations have produced data for graphs with small numbers of vertices, but a general pattern has not been discerned. Part of the problem is that it can be very hard to tell if two graphs are isomorphic. Even using high speed computers, the number of cases that need to be checked, even for relatively small graphs, can require excessive amounts of computing time. To see why, note that for a graph with 20 vertices, there are $20!=2432902008176640000$ permutations of the vertices, which does not even include the possible rearrangements of the edge set. In addition, isomorphic graphs can look quite different. For example, in Figure 4 the two circulant graphs appear to be different but are actually isomorphic circulant graphs. A formula or mathematical shortcut to provide information on the number of non-isomorphic circulant graphs of order $n$ would be useful, even if only a partial result or pattern is obtained.


Figure 4. The circulant graphs $\mathrm{C}[11,\{2,3,4\}]$ and $\mathrm{C}[11,\{1,3,5\}]$.
Figure 4. $\mathrm{C}[11,\{2,3,4\}]$ and $\mathrm{C}[11,\{1,3,5\}]$ look quite different but are actually isomorphic, since there is a known permutation of the vertices of the first graph which produces the second graph and all edge adjacencies.

### 1.4 Circulant Linear System Of Equations

Circulant behavior is a common theme in mathematics, so it is not unexpected that this research can be applied to problems from linear algebra. A circulant system of linear equations consists of $n$ equations and $n$ unknowns where each variable is algebraically linked to the same set of relative variables. Note that this is basically the same way circulant graphs are defined. In the linear system, variables play the same role as the vertices in a circulant graph. The concept of "keys" can even be used the same way for both cases.


Figure 5: A circulant linear system with 9 variables and its corresponding graph C[9,\{3\}]
Above is a circulant system of nine equations where each variable is linked to the 3rd variable to the right. Once the end of the list is reached, one "wraps" around to the beginning of the variable list. Notice that this system decomposes into three independent subsystems involving $\left\{\mathrm{x}_{1}, \mathrm{x}_{4}, \mathrm{x}_{7}\right\},\left\{\mathrm{x}_{2}, \mathrm{x}_{5}, \mathrm{x}_{8}\right\}$, and $\left\{\mathrm{x}_{3}, \mathrm{x}_{6}, \mathrm{x}_{9}\right\}$. The graph on the right is the circulant graph with nine vertices where each vertex is connected to the third vertex over, i.e. C[9,\{3\}]. Note that this graph decomposes into three isomorphic connected components, each containing three nodes. Hence, this graph models the combinatorial aspects of the systems algebraic linkages. Circulant linear systems of homogeneous equations have applications in discrete dynamical systems. Further understanding of the solution sets for these systems could provide useful information for the long term behavior of these dynamical systems ${ }^{4}$. If the combinatorics of these linear systems are better understood and reveal information about their solutions sets over finite fields, expanded knowledge about transitory and attractor portions of the corresponding dynamical systems will follow.

## 2. Results

Whether or not two circulant graphs are isomorphic can easily be determined if the behavior of each graph is known. As a result of this research project, there are now five theorems that will determine the behavior of a circulant graph, with one key, no matter how complicated. Specifically, the four lemmas below help determine how the circulant graph will decompose, whether into independent cycles or one continuous cycle. Knowing how the graph decomposes also determines whether or not the solution set for the corresponding circulant system of equations would be trivial. The last theorem examines how the circulant system of linear equations behaves, and the four lemmas can be used to solidify the conclusion as to whether or not the circulant system of linear equations has only the zero solution or has more than just the zero solution.

Lemma 2.1 Let $m,(d+1) \in \mathbf{Z}$, such that, m is the order and $(d+1)$ is the distance between the vertices. The circulant graph contains $(d+1)$ independent cycles, each with an even number of edges, if and only if, $\frac{m}{\operatorname{gcd}((d+1), m)}$ is even.

Lemma 2.2 1 Let $m,(d+1) \in \mathbf{Z}$, such that, $m$ is the order and $(d+1)$ is the distance between the vertices. The circulant graph contains $(d+1)$ independent cycles, each with an odd number of edges, if and only if, $\frac{m}{\operatorname{gcd}((d+1), m)}$ is odd and (d+1) $\mid$ m.

Lemma 2.3 Let $m,(d+1) \in \mathbf{Z}$, such that, $m$ is the order and $(d+1)$ is the distance between the vertices. The circulant graph contains $\frac{(d+1)}{2}$ independent cycles, each with an odd number of edges, if and only if, $\frac{m}{\operatorname{gcd}((d+1), m)}$ is even, $(d+1) \geq 4$, and $\frac{(d+1)}{2}$ is not divisible by m .

Lemma 2.4 Let $m,(d+1) \in \mathbf{Z}$, such that, $m$ is the order and $(d+1)$ is the distance between the vertices. The circulant graph contains one continuous cycle, if and only if, $\frac{m}{\operatorname{gcd}((d+1), m)}$ is odd and $(d+1)$ is not divisible by m .

Theorem 2.5 Let $m, n,(d+1) \in \mathbf{Z}$, such that, $m$ is the order, $n$ is the modulus, and $(d+1)$ is the distance between the vertices. The homogeneous circulant system of linear equations contains only the zero solution, if and only if, the $n$ is odd and $\frac{m}{\operatorname{gcd}(d+1, m)}$ is odd.

Proof. The contrapositive of Theorem 2.5 reads.

$$
=>: \text { Assume that the modulus } n \text { or } \frac{m}{\operatorname{gcd}(d+1, m)} \text { is even. }
$$

First, let $n \in \mathbf{Z}$, such that, $n$ is even. If $n$ is even, then the solutions of the circulant system of linear equations, for any number of variables, $m$, will always contain the zero solution: $(0,0, \ldots, 0)$ and some non-zero solution: ( $\mathrm{n} / 2, \mathrm{n} / 2$, $\ldots, \mathrm{n} / 2$ ). Thus, the circulant system of linear equations has more than the zero solution when n is even.

Next, let $\frac{m}{\operatorname{gcd}(d+1, m)}$ be even. We have two cases where this is true.
Case 1 By Lemma 2.1, the circulant graph will separate into $(d+1)$ independent cycles, each with an even number of edges. This means that the corresponding circulant system of linear equations will separate into $(d+1)$ subsystems of linear equations; $x_{1}=-x_{(d+2)}=\ldots=-x_{(m-d)}=x_{1}, x_{2}=-x_{(d+3)}=\ldots=-x_{m-(d-1)}=x_{2}, \ldots, x_{(d+1)}=-x_{2(d+1)}=\ldots=-x_{m}=x_{(d+1)}$.

.
-
$\mathrm{X}_{(\mathrm{d}+1)}+\mathrm{x}_{\mathrm{m}}=0$


Case 2 By Lemma 2.3, the circulant graph will separate into $\frac{(d+1)}{2}$ independent cycles, each with an odd number of edges. This means that the corresponding circulant system of linear equations will separate into $\frac{(d+1)}{2}$ subsystems of linear equations; $x_{1}=-x_{(d+2)}=\ldots=x_{(m-d)}=-x_{1}, x_{2}=-x_{(d+3)}=\ldots=x_{m-(d-1)}=-x_{2}, \ldots, x_{((d+1) / 2)}=-x_{2 d}=\ldots=x_{2(d+1)}=-$ $\mathrm{X}_{((\mathrm{d}+1) / 2)}$.

$$
\begin{aligned}
& \begin{array}{cll}
\mathrm{x}_{1}+\mathrm{X}_{(\mathrm{d}+2)} & =0 & \mathrm{x}_{1}=-\mathrm{x}_{(\mathrm{d}+2)}=\ldots=\mathrm{x}_{(\mathrm{m}-\mathrm{d})}=-\mathrm{X}_{1} \\
\mathrm{x}_{2}+\mathrm{X}_{(\mathrm{d}+3)} & =0 & \mathrm{x}_{2}=-\mathrm{x}_{(\mathrm{d}+3)}=\ldots=\mathrm{x}_{\mathrm{m}-(\mathrm{d}-1)}=-\mathrm{x}_{2}
\end{array} \\
& x_{3}+x_{(d+4)}=0 \\
& \text { • } \\
& \mathrm{x}_{4} \quad+\mathrm{x}_{(\mathrm{d}+5)}=0 \\
& \text { • } \\
& \text { • } \\
& \mathrm{X}_{((\mathrm{d}+1) / 2)}=-\mathrm{X}_{2 \mathrm{~d}}=\ldots=\mathrm{X}_{2(\mathrm{~d}+1)}=-\mathrm{X}_{((\mathrm{d}+1) / 2)} \\
& \mathrm{X}_{1}+\mathrm{X}_{(\mathrm{m}-\mathrm{d})} \quad=0 \\
& \cdot \\
& \text { • } \\
& \mathrm{X}_{(\mathrm{d}+1)}+\mathrm{X}_{\mathrm{m}}=0
\end{aligned}
$$



Each subsystem contains $n$ amount of solutions over mod $n$, both zero: $(0,0, \ldots, 0)$ and non-zero: $\left(1,1, \ldots, 1_{(d+1)}\right.$, $\left.(\mathrm{n}-1),(\mathrm{n}-1), \ldots,(\mathrm{n}-1)_{(\mathrm{d}+1)}\right)$ or $\left(1,1, \ldots, 1_{((\mathrm{d}+1) / 2)},(\mathrm{n}-1),(\mathrm{n}-1), \ldots,(\mathrm{n}-1)_{((\mathrm{d}+1) / 2)}\right)$, regardless of the orientation of n . Thus, the circulant system of linear equations has more than the zero solution when $\frac{m}{\operatorname{gcd}(d+1, m)}$ is even.

The reader can check that if the homogeneous circulant system of equations contains more than the zero solution, then $n$ is even or $\frac{m}{\operatorname{gcd}(d+1, m)}$ is even.

It is always the case, that when $n$ is even for any number of variables, $m$, the solution set consists of the zero solution: $(0,0, \ldots, 0)$ and non-zero solutions: $(\mathrm{n} / 2, \mathrm{n} / 2, \ldots, \mathrm{n} / 2)$. It is also always the case that when $\frac{m}{\operatorname{gcd}(d+1, m)}$ is even, the homogenous circulant systems of linear equations decomposes into $(\mathrm{d}+1)$ or $\frac{(d+1)}{2}$ independent subsystems of linear equations each with the zero and non-zero solutions. Thus, if $n$ and $\frac{m}{\operatorname{gcd}(d+1, m)}$ are even, then the homogeneous circulant system of linear equations contains more than the zero solution.

Therefore, by Proof by Contrapositive, a homogeneous circulant system of linear equations contains only the zero solution, if and only if, $n$ and $\frac{m}{\operatorname{gcd}(d+1, m)}$ are odd.

## 3. Conclusion

There are now number theoretic formulas that describe how some circulant graphs decompose into subsystems, which can also be related to the graph's corresponding homogenous linear equations because they provide useful information regarding the solution sets of said equations. The lemmas can be used to decompose circulant graphs, which will simplify the work needed to determine whether two circulant graphs are isomorphic.

The method involving searching for patterns in predetermined isomorphic circulant graph data by Dr. Kehle was inconclusive. In order to find anything of significance, we would need extensive amounts of computer time to develop more data, which we did not have.

In the future, we plan to continue to look for patterns in the existing non-isomorphic graph data. We will also find general theorems for the behavior of circulant linear systems of equations and their corresponding circulant graphs when there are two or more keys present. This concept will also be applied to discrete dynamical systems and how they decompose. Specifically, we will be investigating cellular automata, which are a finite dynamical system who share the same properties as homogenous circulant linear systems.

## 4. Acknowledgements

We would like to thank the Henson School of Science and Technology and the National Science Foundation and their Bridges for SUCCESS (Salisbury University's Connections to Careers for Every STEM Student) program for supporting our research; which is funded by the National Science Foundation's STEM Talent Enhancement Program (STEP), award number 0969428.

Special thanks to Dr. Paul Kehle for providing us with the needed existing non - isomorphic graphs per order data sheet.

## 5. References

[1] Gosset, Eric. Discrete Mathematics with Proof. Pearson Education, Inc. 2003. Print.
[2] Kehle, Paul. Mathematical Experiments and Experimental Mathematics. Consortium, Number 98, Spring/Summer 2010. pp. 22-33. Print.
[3] Marcus, Daniel A. Graph Theory: A Problem Oriented Approach. MAA Textbooks, 2008. Print.
[4] Seaborn, Joseph; Churchill, Aaron; Mummert, Pillip; Bardzell, Michael. The Evolution of Finite 1-
Dimensional Cellular Automata updated with k-Rules, Journal of Cellular Automata. Volume 6, Number 6. 2011. pp. 505-515. Print.
[5] Goodman, Len; Lauschke, Andreas; and Weisstein, Eric W. "Dijkstra's Algorithm." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/DijkstrasAlgorithm.html
[6] Weisstein, Eric W. "Directed Graph." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/DirectedGraph.html

