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Investigating Cardano's Irreducible Case

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Abstract

Solving cubic equations is a historically rich problem in mathematics. Unlike with quadratic equations, cubic equations do not have a simple "cubic formula" when considering only real numbers. However, over the years many techniques have been presented that often find the solutions of cubic equations. This research investigates one of these techniques known as Cardano's Method. This method provides an algebraic technique for solving the general cubic equation. Since its inception, this technique has suffered a significant drawback. In some instances, the application of Cardano's Method results in what Cardano termed the "irreducible case." The irreducible case occurs when a complex number is needed to complete the process. This research investigates the relationship among the coefficients of the general cubic equation and the irreducible case, and it has determined that these relationships fall into one of three categories: always reducible, always irreducible, or conditionally irreducible. This research has discovered which relationships fall into each of the aforementioned categories. One possible result of this research is the formulation of a general algorithm to easily determine whether a given cubic equation will produce Cardano's irreducible case.

Keywords: Cardano, Cubic, Irreducible

1. Introduction

In 1545, Gerolamo Cardano published in his *Ars Magna* a method for solving for the roots of cubic equations. This technique, now known as Cardano's Method, was originally derived by Scipione del Ferro of Bologna. The Venetian mathematician Tartaglia was able to re-derive del Ferro's technique after he was challenged to a mathematical duel by del Ferro's apprentice. Eventually Tartaglia divulged his method to Cardano with the condition of not publishing his findings. Cardano, who later discovered del Ferro's notes on the method at the University of Bologna, reneged on this agreement and published Tartaglia's work.

Cardano's Method solves for the roots of a general cubic equation by transforming the cubic into what is known as a depressed cubic equation. A depressed cubic lacks the quadratic term. In this form, the equation is then related to a geometric representation of a cube, and the volume of this cube is used to solve for one of the roots of the original cubic equation by using the quadratic formula. With one root found, the other two roots of the cubic equation can be discovered by factoring out the root found through Cardano's Method from the original cubic equation and then applying the quadratic formula to the resulting quadratic equation.[1]

Cardano's Method does have a problem that occurs when applying the quadratic formula within the method. If the value of the discriminant is negative, then complex numbers are produced. This is known as Cardano's irreducible case. The irreducible case can occur even when all roots of the cubic are real numbers. This research determines when the irreducible case will occur without needing to directly apply Cardano's Method. Coefficient relationships are examined to determine if Cardano's Method can be applied effectively before executing the method itself.

2. Methodology

It is beneficial to give a brief overview of how Cardano's Method is derived. The method begins with the general cubic equation (1). A substitution (2) is applied for the variable, which simplifies the equation to the depressed cubic form (3). It should be noted that the coefficients of the depressed cubic are defined in terms of the coefficients of the original cubic as seen in (4) and (5).

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

$$x = u - \frac{b}{3a} \tag{2}$$

$$u^3 + mu = n \tag{3}$$

$$m = \frac{c}{a} - \frac{b^2}{3a^2} \tag{4}$$

$$n = -\frac{2b^3}{27a^3} + \frac{bc}{3a^2} - \frac{d}{a} \tag{5}$$

Here Cardano uses a geometric representation for a cube to solve for the roots of the cubic equation. The cube is divided into six partitions: two cubes and four rectangular prisms (Figure 1).

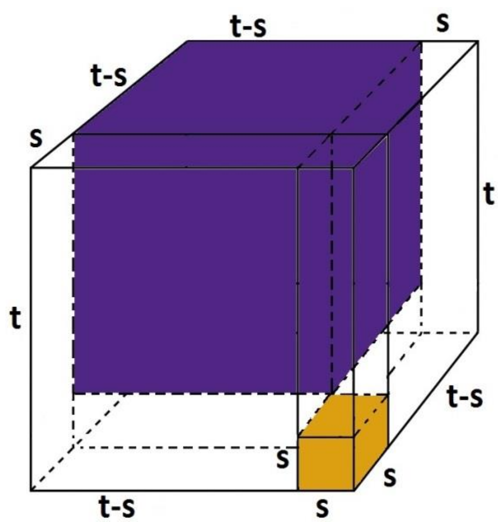

Figure 1. Cardano's geometric representation of a cube with partitions

The volume of this cube is determined by summing the volumes of the partitions (6).

$$t^3 = s^3 + s^2(t-s) + 2st(t-s) + s(t-s)^2 + (t-s)^3 \tag{6}$$

$$t^3 - s^3 = (t-s)[(t-s)^2 + 3st] \tag{7}$$

$$u = t - s \tag{8}$$

$$u^3 + 3stu = t^3 - s^3 \tag{9}$$

Equation (6) is simplified into the form of equation (7). By considering the variable *u*, defined in equation (8), manipulations are made to equation (7) to form the depressed cubic as seen in equation (9). Then the terms *t* and *s* are related to the coefficients *m* and *n* in equation (3). The term *s* in equation (10) is then substituted into *s* in equation (11). Then equation (11) is multiplied by t^3 to form the quadratic equation in terms of t^3 seen in equation (12). The quadratic formula is applied to equation (12), and then the cubic root is taken to form equation (13).

$$s = \frac{m}{3t} \tag{10}$$

$$n = t^3 - s^3 \tag{11}$$

$$0 = t^6 - nt^3 - \frac{1}{27}m^3 \tag{12}$$

$$t = \sqrt[3]{\frac{n \pm \sqrt{n^2 + \frac{4}{27}m^3}}{2}} \tag{13}$$

Once *t* is found, equation (11) is used to find *s*. Then equation (8) is used to find *u*. Finally, equation (2) is used to determine *x*. Now that one of the roots of the original cubic equation is known, the other two can be found by factoring that root out of the cubic and applying the quadratic formula to the resulting quadratic equation.[2]

It should be noted that part of the solution for *t* involves a square root with discriminant (14). Since real numbers are produced by the square root function only when the discriminant is nonnegative, Cardano's irreducible case occurs when that discriminant is negative. That is, a cubic equation is irreducible whenever (14) is less than zero.[3] To determine what causes this to occur, the values of *m* and *n* must be considered.

$$n^2 + \frac{4}{27}m^3 \tag{14}$$

The initial approach to determine when Cardano's irreducible case occurs focused exclusively on analyzing *m* and *n*, which saw varying degrees of success. Thus, the approach was changed to look specifically at the values of the original coefficients *a*, *b*, *c*, and *d*. To simplify matters, only the case where $a = 1$ is examined. Since *a* is nonzero for any cubic equation, it is always possible to divide through by the value of *a* to arrive at this special case. This further simplifies the definitions of *m* (15) and *n* (16). These expressions can be used to fully expand and simplify the original discriminant to the form (17).

$$m = c - \frac{1}{3}b^2 \tag{15}$$

$$n = -\frac{2}{27}b^3 + \frac{1}{3}bc - d \tag{16}$$

$$n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3 d - \frac{1}{27}b^2 c^2 - \frac{2}{3}bcd + \frac{4}{27}c^3 + d^2 \tag{17}$$

To continue the analysis, Table 1 was constructed with all possible sign values of *b*, *c*, and *d*. Each case was examined and the behavior of the discriminant (17) under the given conditions was analyzed.

Table 1. all 27 relations of *b*, *c*, and *d*, given *a* = 1

Case	b	c	d	Case	b	c	d	Case	b	c	d
1	0	0	0	10	+	0	0	19	-	0	0
2	0	0	+	11	+	0	+	20	-	0	+
3	0	0	-	12	+	0	-	21	-	0	-
4	0	+	0	13	+	+	0	22	-	+	0
5	0	+	+	14	+	+	+	23	-	+	+
6	0	+	-	15	+	+	-	24	-	+	-
7	0	-	0	16	+	-	0	25	-	-	0
8	0	-	+	17	+	-	+	26	-	-	+
9	0	-	-	18	+	-	-	27	-	-	-

3. Findings

Analysis of each case leads to a conclusion that may belong to one of three categories. If a case *always* results in the irreducible case, it is called irreducible. If a case *never* results in the irreducible case, it is called reducible. However, if a case must satisfy some *condition* to result in the irreducible case, it is called conditionally irreducible. The following section includes discussion for all 27 cases. Each case is classified accordingly as irreducible, reducible, or conditionally irreducible.

3.1. cases 1 – 9: $b = 0$

3.1.1. case 1($b=0, c=0, d=0$ *) is reducible*

Proof:
If $b=0, c=0,$ and $d=0$, then $m=0$ and $n=0$. Hence, $n^2+\frac{4}{27}m^3=0$. Therefore, Case 1 is reducible. □

3.1.2. case 2 ($b=0, c=0, d>0$ *) is reducible*

Proof:
If $b = 0$, $c = 0$, and $d > 0$, then $m = 0$ and $n=-d$. Hence, $n^2+\frac{4}{27}m^3=d^2>0$. Therefore, Case 2 is reducible. □

3.1.3. case 3 ($b=0, c=0, d<0$ *) is reducible*

Proof:
If $b = 0$, $c = 0$, and $d < 0$, then $m = 0$ and $n=-d$. Hence, $n^2+\frac{4}{27}m^3=d^2>0$. Therefore, Case 3 is reducible. □

3.1.4. case 4 ($b=0, c>0, d=0$ *) is reducible*

Proof:
If $b = 0$, $c > 0$, and $d = 0$, then $m = c$ and $n = 0$. Hence, $n^2+\frac{4}{27}m^3=\frac{4}{27}c^3.$ Since $c > 0$, $\frac{4}{27}c^3>0.$ So, $n^2+\frac{4}{27}m^3>0.$ Therefore, Case 4 is reducible. □

3.1.5. case 5 ($b = 0, c > 0, d > 0$) is reducible

Proof:
If $b = 0$, $c > 0$, and $d > 0$, then $m = c$ and $n = -d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}c^3 + d^2$. Since $c > 0$, $\frac{4}{27}c^3 + d^2 > 0$. So, $n^2 + \frac{4}{27}m^3 > 0$. Therefore, Case 5 is reducible. □

3.1.6. case 6 ($b = 0, c > 0, d < 0$) is reducible

Proof:
If $b = 0$, $c > 0$, and $d < 0$, then $m = c$ and $n = -d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}c^3 + d^2$. Since $c > 0$, $\frac{4}{27}c^3 + d^2 > 0$. So, $n^2 + \frac{4}{27}m^3 > 0$. Therefore, Case 6 is reducible. □

3.1.7. case 7 ($b = 0, c < 0, d = 0$) is irreducible

Proof:
If $b = 0$, $c < 0$, and $d = 0$, then $m = c$ and $n = 0$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}c^3$. Since $c < 0$, $\frac{4}{27}c^3 < 0$. So, $n^2 + \frac{4}{27}m^3 < 0$. Therefore, Case 7 is irreducible. □

3.1.8. case 8 ($b = 0, c < 0, d > 0$) is conditionally irreducible

Proof:
If $b = 0$, $c < 0$, and $d > 0$, then $m = c$ and $n = -d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}c^3 + d^2$. So, Case 8 is irreducible if $\frac{4}{27}c^3 + d^2 < 0$. Therefore, Case 8 is conditional and is irreducible whenever $27d^2 < -4c^3$. □

3.1.9. case 9 ($b = 0, c < 0, d < 0$) is conditionally irreducible

Proof:
If $b = 0$, $c < 0$, and $d < 0$, then $m = c$ and $n = -d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}c^3 + d^2$. So, Case 9 is irreducible if $\frac{4}{27}c^3 + d^2 < 0$. Therefore, Case 9 is conditional and is irreducible whenever $27d^2 < -4c^3$. □

3.2. cases 10 – 18: $b > 0$

3.2.1. case 10 ($b > 0, c = 0, d = 0$) is reducible

Proof:
If $b > 0$, $c = 0$, and $d = 0$, then $m = -\frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3$. Hence, $n^2 + \frac{4}{27}m^3 = 0$. Therefore, Case 10 is reducible.□

3.2.2. case 11 ($b > 0, c = 0, d > 0$) is reducible

Proof:
If $b > 0$, $c = 0$, and $d > 0$, then $m = -\frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 - d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3 d + d^2$. Since $b > 0$ and $d > 0$, $\frac{4}{27}b^3 d + d^2 > 0$. So, $n^2 + \frac{4}{27}m^3 > 0$. Therefore, Case 11 is reducible. □

3.2.3 *case 12 ($b > 0, c = 0, d < 0$) is conditionally irreducible*

Proof:
If $b > 0$, $c = 0$, and $d < 0$, then $m = -\frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 - d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3 d + d^2 = d\left(\frac{4}{27}b^3 + d\right)$. Since $d < 0$, Case 12 is irreducible if $\frac{4}{27}b^3 + d > 0.$ Therefore, Case 12 is conditional and is irreducible whenever $27d > -4b^3$. □

3.2.4. *case 13 ($b > 0, c > 0, d = 0$) is conditionally irreducible*

Proof:
If $b > 0$, $c > 0$, and $d = 0$, then $m = c - \frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 + \frac{1}{3}bc$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 = \frac{1}{27}c^2(4c - b^2).$ So, Case 13 is irreducible if $4c - b^2 < 0.$ Therefore, Case 13 is conditional and is irreducible whenever $b^2 > 4c$. □

3.2.5. *case 14 ($b > 0, c > 0, d > 0$) is conditionally irreducible*

Proof:
If $b > 0$, $c > 0$, and $d > 0$, then $m = c - \frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 + \frac{1}{3}bc - d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3 d - \frac{2}{3}bcd + \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 + d^2$. So, Case 14 is irreducible if $\frac{4}{27}b^3 d - \frac{2}{3}bcd + \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 + d^2 < 0$. Therefore, Case 14 is conditional and is irreducible whenever $4c^3 + 4b^3 d + 27d^2 < b^2c^2 + 18bcd$. □

3.2.6 *case 15 ($b > 0, c > 0, d < 0$) is conditionally irreducible*

Proof:
If $b > 0$, $c > 0$, and $d < 0$, then $m = c - \frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 + \frac{1}{3}bc - d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3 d - \frac{2}{3}bcd + \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 + d^2$. So, Case 15 is irreducible if $\frac{4}{27}b^3 d - \frac{2}{3}bcd + \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 + d^2 < 0$. Therefore, Case 15 is conditional and is irreducible whenever $4c^3 + 4b^3 d + 27d^2 < b^2c^2 + 18bcd$. □

3.2.7. *case 16 ($b > 0, c < 0, d = 0$) is irreducible*

Proof:
If $b > 0$, $c < 0$, and $d = 0$, then $m = c - \frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 + \frac{1}{3}bc$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}c^3 - \frac{1}{27}b^2c^2$. Since $c < 0$, $\frac{4}{27}c^3 - \frac{1}{27}b^2c^2 < 0.$ So, $n^2 + \frac{4}{27}m^3 < 0.$ Therefore, Case 16 is irreducible. □

3.2.8. *case 17 ($b > 0, c < 0, d > 0$) is conditionally irreducible*

Proof:
If $b > 0$, $c < 0$, and $d > 0$, then $m = c - \frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 + \frac{1}{3}bc - d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3 d - \frac{2}{3}bcd + \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 + d^2$. So, Case 17 is irreducible if

$\frac{4}{27}b^3d - \frac{2}{3}bcd + \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 + d^2 < 0$. Therefore, Case 17 is conditional and is irreducible whenever $4c^3 + 4b^3d + 27d^2 < b^2c^2 + 18bcd$. □

3.2.9. case 18 ($b > 0, c < 0, d < 0$) is conditionally irreducible

<u>Proof</u>:
If $b > 0$, $c < 0$, and $d < 0$, then $m = c - \frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 + \frac{1}{3}bc - d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3d - \frac{2}{3}bcd + \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 + d^2$. So, Case 18 is irreducible if $\frac{4}{27}b^3d - \frac{2}{3}bcd + \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 + d^2 < 0$. Therefore, Case 18 is conditional and is irreducible whenever $4c^3 + 4b^3d + 27d^2 < b^2c^2 + 18bcd$. □

3.3. cases 19 – 27: $b < 0$

3.3.1. case 19 ($b < 0, c = 0, d = 0$) is reducible

<u>Proof</u>:
If $b < 0$, $c = 0$, and $d = 0$, then $m = -\frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3$. Hence, $n^2 + \frac{4}{27}m^3 = 0$. Therefore, Case 19 is reducible.□

3.3.2. case 20 ($b < 0, c = 0, d > 0$) is conditionally irreducible

<u>Proof</u>:
If $b < 0$, $c = 0$, and $d > 0$, then $m = -\frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 - d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3d + d^2 = d\left(\frac{4}{27}b^3 + d\right)$. Since $d < 0$, Case 20 is irreducible if $\frac{4}{27}b^3 + d > 0$. Therefore, Case 20 is conditional and is irreducible whenever $27d > -4b^3$. □

3.3.3. case 21 ($b < 0, c = 0, d < 0$) is reducible

<u>Proof</u>:
If $b < 0$, $c = 0$, and $d < 0$, then $m = -\frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 - d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3d + d^2$. Since $b < 0$ and $d < 0$, $\frac{4}{27}b^3d + d^2 > 0$. So, $n^2 + \frac{4}{27}m^3 > 0$. Therefore, Case 21 is reducible. □

3.3.4. case 22 ($b < 0, c > 0, d = 0$) is conditionally irreducible

<u>Proof</u>:
If $b < 0$, $c > 0$, and $d = 0$, then $m = c - \frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 + \frac{1}{3}bc$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 = \frac{1}{27}c^2(4c - b^2)$. So, Case 22 is irreducible if $4c - b^2 < 0$. Therefore, Case 22 is conditional and is irreducible whenever $b^2 > 4c$. □

3.3.5. case 23 ($b < 0, c > 0, d > 0$) is conditionally irreducible

<u>Proof</u>:
If $b < 0$, $c > 0$, and $d > 0$, then $m = c - \frac{1}{3}b^2$ and $n = -\frac{2}{27}b^3 + \frac{1}{3}bc - d$. Hence, $n^2 + \frac{4}{27}m^3 = \frac{4}{27}b^3d - \frac{2}{3}bcd + \frac{4}{27}c^3 - \frac{1}{27}b^2c^2 + d^2$. So, Case 23 is irreducible if

$\frac{4}{27}b^3d-\frac{2}{3}bcd+\frac{4}{27}c^3-\frac{1}{27}b^2c^2+d^2<0$. Therefore, Case 23 is conditional and is irreducible whenever $4c^3+4b^3d+27d^2<b^2c^2+18bcd$. □

3.3.6. *case 24 ($b<0, c>0, d<0$) is conditionally irreducible*

Proof:
If $b < 0$, $c > 0$, and $d < 0$, then $m=c-\frac{1}{3}b^2$ and $n=-\frac{2}{27}b^3+\frac{1}{3}bc-d$. Hence, $n^2+\frac{4}{27}m^3=\frac{4}{27}b^3d-\frac{2}{3}bcd+\frac{4}{27}c^3-\frac{1}{27}b^2c^2+d^2$. So, Case 24 is irreducible if $\frac{4}{27}b^3d-\frac{2}{3}bcd+\frac{4}{27}c^3-\frac{1}{27}b^2c^2+d^2<0$. Therefore, Case 24 is conditional and is irreducible whenever $4c^3+4b^3d+27d^2<b^2c^2+18bcd$. □

3.3.7. *case 25 ($b<0, c<0, d=0$) is irreducible*

Proof:
If $b < 0$, $c < 0$, and $d = 0$, then $m=c-\frac{1}{3}b^2$ and $n=-\frac{2}{27}b^3+\frac{1}{3}bc$. Hence, $n^2+\frac{4}{27}m^3=\frac{4}{27}c^3-\frac{1}{27}b^2c^2$. Since $c < 0$, $\frac{4}{27}c^3-\frac{1}{27}b^2c^2<0$. So, $n^2+\frac{4}{27}m^3<0$. Therefore, Case 25 is irreducible. □

3.3.8. *case 26 ($b<0, c<0, d>0$) is conditionally irreducible*

Proof:
If $b < 0$, $c < 0$, and $d > 0$, then $m=c-\frac{1}{3}b^2$ and $n=-\frac{2}{27}b^3+\frac{1}{3}bc-d$. Hence, $n^2+\frac{4}{27}m^3=\frac{4}{27}b^3d-\frac{2}{3}bcd+\frac{4}{27}c^3-\frac{1}{27}b^2c^2+d^2$. So, Case 26 is irreducible if $\frac{4}{27}b^3d-\frac{2}{3}bcd+\frac{4}{27}c^3-\frac{1}{27}b^2c^2+d^2<0$. Therefore, Case 26 is conditional and is irreducible whenever $4c^3+4b^3d+27d^2<b^2c^2+18bcd$. □

3.3.9. *case 27 ($b<0, c<0, d<0$) is conditionally irreducible*

Proof:
If $b < 0$, $c < 0$, and $d < 0$, then $m=c-\frac{1}{3}b^2$ and $n=-\frac{2}{27}b^3+\frac{1}{3}bc-d$. Hence, $n^2+\frac{4}{27}m^3=\frac{4}{27}b^3d-\frac{2}{3}bcd+\frac{4}{27}c^3-\frac{1}{27}b^2c^2+d^2$. So, Case 27 is irreducible if $\frac{4}{27}b^3d-\frac{2}{3}bcd+\frac{4}{27}c^3-\frac{1}{27}b^2c^2+d^2<0$. Therefore, Case 27 is conditional and is irreducible whenever $4c^3+4b^3d+27d^2<b^2c^2+18bcd$. □

4. Conclusion and Directions for Future Research

This analysis has discovered that 37% of the cases are always reducible, 11% of the cases are always irreducible, and 52% of the cases are conditionally irreducible. Further analysis of the findings may identify simple relationships between the cases and may lead to a more efficient identification of whether Cardano's irreducible case will occur. For example, according to Cases 1-6, it is clear that the irreducible case will not occur when *b* is zero and *c* is nonnegative. Relationships like this could make identification of the occurrence of Cardano's irreducible case as simple as a quick glance at the cubic equation of interest. This information may be valuable in computer programs that implement Cardano's Method to solve for the roots of cubic equations. Moreover, an algorithm can easily be constructed from these findings to determine if a given cubic equation's coefficients will cause the irreducible case.

Although some additional work is necessary to fully simplify the findings for the conditional results, this work has identified one approach to analyzing and describing all possible coefficient relationships of cubic equations that will or will not cause Cardano's irreducible case.

5. References

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