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Strengthen Elementary Students' Understanding of Factors

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Abstract

Research on pre-service elementary school teachers' understanding of the multiplicative structure of the natural numbers demonstrates an under-utilization of unique prime factorization in the identification of a number's factors. For example, Zazkis and Campbell found that a majority of teacher candidates employed trial division to analyze factor-candidates of a number, even when both were presented in prime-factored-form. [1] Recent studies have shown that teachers' understanding of factor can be strengthened by engaging in a sequence of instructional tasks that explore the relationship between a number's prime factorization and its factors.[2] This study seeks to extend the scope of investigation in this area to the population of elementary school students. The research questions addressed by the study are:

1. To what extent do elementary school students under-utilize unique prime factorization in the identification of a number's factors?
2. How is elementary students' usage of unique prime factorization in the identification of a number's factors similar to, or different from, that of pre-service elementary school teachers as identified in the research literature?
3. Which instructional tasks strengthen elementary school students' understanding of the use of prime factorization in the identification of a number's factors?

Researchers conducted teaching experiments in two elementary school mathematics classrooms. A mix-methods analysis of quantitative pre- and post-test data and qualitative student work was employed.

Keywords: Factorization, Prime Factors, Education

1. Introduction

Consider the equation N=2x3x5x19 and the set of numbers: {5, 11, 38, 20}. Which of these numbers are factors of N? One way to think of this problem is to compute N, and trial divide the four numbers. Without any computation errors, you would be able to successfully identify the factors through the understanding of the definition of a factor: a number would be a factor of N if it evenly divides N. Using this method, you could successfully identify that 5 and 38 are both factors of N, while 11 and 20 are not. This trial division method is the one most often taught in schools, and, arguably, the one most likely employed by the average person faced with solving this problem.

The second method of approaching this problem is through the use of prime factorization. This method employs a richer understanding of divisibility and factor concepts, and is less prone to computation errors. An important result in number theory, known as the Fundamental Theorem of Arithmetic (FTA), tells us that each natural number excluding 1, any number of the set {2, 3, 4, …}, is *uniquely* determined by its prime factorization. No two natural numbers share the same prime factorization; it can be thought of as a number's "DNA." In this instance, N has been presented in its prime factor form: all of the numbers multiplied together are primes. By the FTA, N's factors can only be made up by some combination of the prime numbers 2, 3, 5, and 19. Thus, any number that is not made up of those primes cannot be a factor of N. A factor must only contain primes in the original number, with respect to the total number of each prime. Thinking this way, it is easy to see that 5 is a factor of N; it is in its prime factorization. Similarly, it is straightforward to see that 11 is not a factor of N; nowhere does the prime number 11 exist in N. Moving to think about composite factors requires more information. We must know the prime factorization of the composite numbers to know if they share primes with N. The number 38 is 2x19; since both 2 and 19 are in the prime factorization of N, 38 is a factor of N. The number 20, however, is 2x2x5; N contains the prime factors of both 2 and 5, but the prime factorization of 20 contains two 2's, where N only has one. This means 20 cannot be a factor of N.

This example shows the nuances associated with numbers. In prime factored form, identifying factors and non-factors is easier and less likely to be faulted by conceptual errors compared to trial division. The DNA of the number is available for viewing and can be used easily; it is transparent. However, using this method correctly requires complete understanding of the FTA, specifically the uniqueness of prime factorization, the communicative property of multiplication, and prime and composite numbers. While student in this study were not asked to explicitly explain the FTA, the assessments provide data that show students' increase reliance upon FTA in the analysis of a number's factors. Furthermore, it has been shown that pre-service elementary school teachers (PSTs) struggle to make use of prime factorization and the uniqueness to identify factors,[3] given this, and the lack of research done on elementary school students, there is a gap in learning that needs to be filled. This paper summarizes an intervention conducted with 4th and 5th grade students to develop their abilities in this area.

2. Literature Review

In 1996, Zazkis conducted clinical interviews with 21 PSTs and found that 15 of them exhibited limited and procedural understandings of divisibility.[4] These PSTs admitted to needing to compute the whole number and then trial dividing; this shows a misunderstanding of prime factorization and divisibility. Zazkis supported this finding, showing that PSTs relied on long division or application of divisibility rules with little ability to use prime factorization as a tool for reasoning about factors. [5] Zazkis and Gadowsky demonstrated that PSTs' fail to make use of the *transparent* features of prime factorization. [3] Meaning they did not take into account the transparent nature of numbers represented in prime factored forms. For example, the prime factored representation of N=2x3x5x19, makes the fact that 5 is a factor easily seen, or transparent; PSTs fail to make this connection. Other studies have identified PSTs' misconceptions about factors and prime numbers; such as the notion that bigger numbers have more factors or that prime numbers are small.[1, 3]

Some studies have characterized the extent of PSTs' knowledge of number theory topics. Zazkis found that PSTs use negative descriptions to define prime numbers (e.g., "prime numbers 'cannot be divided', 'cannot be factored' or 'wouldn't have/are not having any other factor'"[6]), which may be an obstacle to achieving a robust conceptual understanding of prime number.[7] Researchers have also noted that PSTs tend to have an easier time identifying factors than non-factors, and are better able to recognize prime factors than composite factors.[1, 4] Zazkis and Campbell noted that PSTs' difficulty with identifying non-factors may be due to a lack of appreciation for the uniqueness feature of the FTA: "Whereas the *existence* of prime decomposition may be taken for granted, the *uniqueness* of prime decomposition appears to be counterintuitive and often a possibility of different prime decompositions is assumed."[8] Liljedahl and colleagues found that the use of a computer program known as *Number Worlds*, which allowed PSTs to experiment with different arrays of the natural numbers, "thickened" student's understandings of factors, multiples, and primes. [9] The authors say, "this use of the adjective 'thick' to describe a learner's layered, rich, contextual, and often affective understanding of a mathematical concept". [10] This study shows that the use of arrays help PSTs to develop a "thicker" understanding of number theory. Roscoe and Feldman provide an intervention with PSTs in a mathematics content university course.[2] They conducted a three-week intervention, with three in-class lessons and two homework assignments. This intervention produced statistically significant results in developing PST's understandings of factors and prime factorization, specifically in identifying prime factors, prime non-factors, composite factors, and composite non-factors. Roscoe and Feldman, Liljedahl, Zazkis, and colleagues' research focuses on PSTs, and not much research has focused on elementary school children. However, the research completed on PSTs is significant because if PSTs do not understand the implications of the FTA, they cannot teach this content to their future students.

Currently, the literature related to elementary school students on this topic is naught. Burkhart has provided an article in *Mathematics Teaching in the Middle School* that provides an intervention possibility without the backup of quantitative data. [11] He explores using actual building blocks to create a visual representation of prime factorizations. This allows students to physically explore the concepts, and transparency, of prime numbers. Burkhart used blocks to help his 6th graders explore how numbers are made up of their unique set of primes.[11] Students then analyze the patterns and structure of the counting numbers to fifty from the appearance of their prime factorizations. Furthermore, Burkhart allowed his students to explore multiplication, division, exponents, factors, multiples, greatest common factors, and least common multiple.[11] While he provides a description of teaching tasks to help students understand these difficult concepts, there is no quantitative evidence of students' gains presented.

3. Methodology

This study is viewed as a teaching experiment aimed at providing students with a robust understanding of the implications of prime factorization as an aid to identifying the factors of a number. This study is an extension of research conducted by Matt Roscoe and Ziv Feldman with PSTs.[2] Our intervention follows a similar course of study; however, ours has been adapted to meet the instructional and developmental needs of 4th and 5th grade students. The study has three research questions:

1. To what extent do elementary school students under-utilize unique prime factorization in the identification of a number's factors?
2. How is elementary students' usage of unique prime factorization in the identification of a number's factors similar to, or different from, that of pre-service elementary school teachers as identified in the research literature?
3. Which instructional tasks strengthen elementary school students' understanding of the use of prime factorization in the identification of a number's factors?

This study's intervention was conducted in two different classrooms, one fourth grade and one fifth grade. The fourth grade classroom completed the intervention in one and a half hour long mathematics lesson. They completed the same worksheets and homework as the fifth graders did. The fifth graders intervention was conducted over two forty-five minute periods. The fourth graders were not familiar with exponents, so prime factorizations were written as such: 8=2x2x2; fifth graders wrote $8=2^3$. As researchers, we also noticed the fifth grade students were more open in their discussion with each other; the discussions from this intervention will obviously vary from class to class. The most important part of this intervention is that there was no overt teaching regarding the ability to find factors using only primes and not trial division. All of the understanding of this concept was derived from student discussion. Students were challenged to look at patterns within the numbers and realize the prime factorization of a number's factors are simply subsets, or groups of, the prime factorization of the number in question. Students were also challenged to look at how to produce the factors of a number when given the number in prime factor form.

The first day of the intervention focused on the necessary definitions and skills needed to fully understand and appreciate prime factorizations of numbers. We began by leading a short discussion about the definitions of prime, composite, and 1. Posters were made which defined prime as “a number whose only factors are 1 and itself”; composite as “a number with more than two factors”; and 1 as “a special number that is neither prime nor composite.” Next, students were lead through the Sieve of Eratosthenes; Eratosthenes was a Greek philosopher who first proposed the method over 2000 years ago. This Sieve is a simple process of elimination that produces all of the prime numbers. See Figure 1 for an explanation of the Sieve.

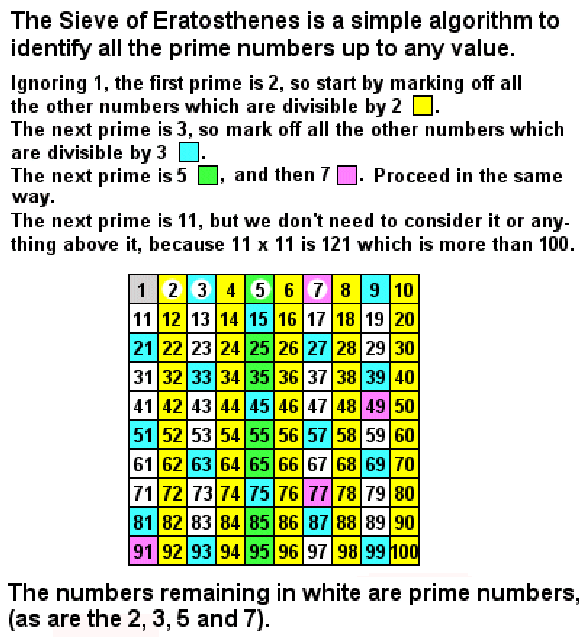

Figure 1: Explanation of the Sieve of Eratosthenes[12]

For our intervention, the fourth grade class completed the Sieve with their classroom teacher, and was observed by one of the researchers to maintain consistency in the intervention. Students were then taught how to make factor trees. Figure 2 shows a sample of the board work done with students to teach this skill. Students were expected to take notes on factor trees and a discussion was lead about if the prime factorization found at the end would be the same no matter how the tree began. Students practiced this skill with a partner on the numbers 36 and 28. One student did one number and the other did the next, and then they taught their partner how to make the factor tree for their number. To finish out the first day, students were split into six groups and completed the factor trees of the numbers 1-50. Students wrote these numbers on index cards, which were taped to a 10 by 5 grid. Table 1 shows the numbers that were assigned to each group. The groupings were based on the difficulty in the factor tree and were assigned to split up the primes as evenly as possible. On every index card, students wrote out the factor tree, identified the number as prime or composite, and wrote out the prime factorization for the number. Once the researcher showed an example, the students worked on completing the chart in their small groups. After completing the grid and discussing the different difficulties of prime factor trees, students were asked to begin the first day’s homework assignment.

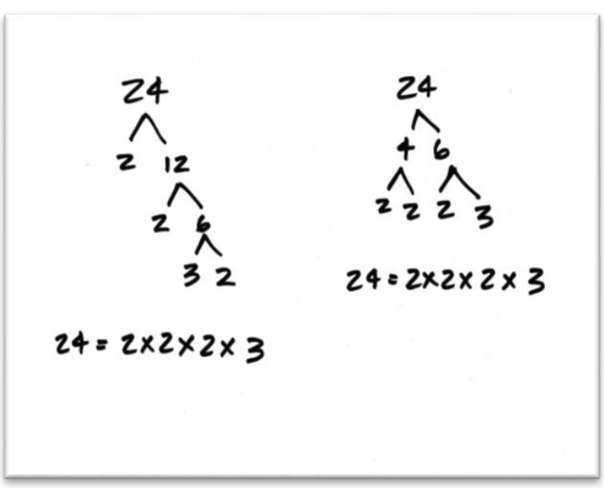

Figure 2: Prime factor trees

Table 1: Day 1 Number Assignments

Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
2	**3**	**5**	**7**	**6**	**1**
15	**14**	**10**	**8**	**13**	**4**
19	**16**	**12**	**9**	**22**	**17**
20	**18**	**26**	**25**	**27**	**21**
34	**23**	**29**	**30**	**36**	**28**
40	**33**	**42**	**31**	**37**	**35**
45	**44**	**43**	**32**	**38**	**41**
47	**49**	**46**	**39**	**50**	**48**

Note: 11 and 24 reserved as teacher examples.

The second day began with a discussion of the homework from the night before. Questions were answered and difficulties discussed (note: the fourth graders had the two days combined). The beginning of the lesson for the second day involved explaining factor pairs to the students. As a class, all the factor pairs for the number 24 were found. Students then paired-up and found the factor pairs for 36 and 28. The most important part of the intervention was when the students found the prime factorization for their number's factors and compared that to the prime factorization for their actual number. They were challenged to look for patterns in the numbers and a discussion followed with the goal to elicit from students that all factors are subsets of a number's prime factorization. Figure 3 shows how the factors of 24, in prime factor form, are subsets of the prime factorization of 24. After this vital discussion was complete on the second day, students were given a chance to begin their homework. The students turned in worksheets from both days as well as completed homework.

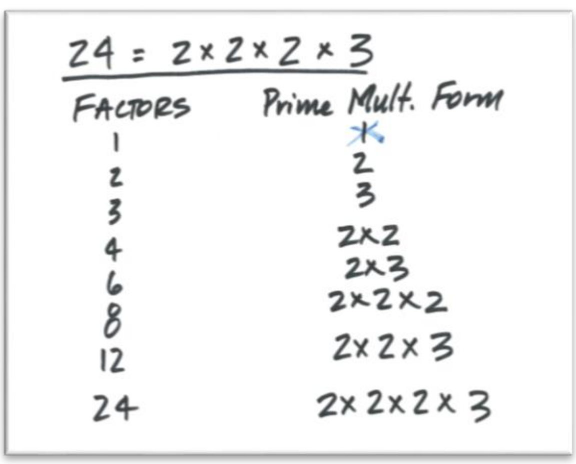

Figure 3: Prime Multiplicative Form Example

Pre- and post-tests were given. The assessments were based on tests given by Zazkis in her research.[4] The pre- and post-test assessments contained the exact same wording with the numbers changed; see table 2 for assessment wording.

Table 2: Assessment

Assessment:

Directions: Answer each question below. Read the directions carefully. Do not use a calculator. Recall that a factor is any number that evenly divides another number. For example, 6 is a factor of 24 but 10 is not a factor of 24.

1. $N = 2\times2\times5\times7\times13$
   a) Is 7 a factor of *N*? How do you know?
   b) Is 3 a factor of *N*? How do you know?
   c) Is 14 a factor of *N*? How do you know?
   d) Is 45 a factor of *N*? How do you know?
2. If $M = 3\times5\times7$ can you find all the factors of M ? Show how you found them.

A rubric was developed to assess the level of change from the pre-assessment to the post-assessment. Both the researchers developed the rubric and conducted a test of inter-rater-reliability. All the scores given to students were agreed upon by both the researchers and the rubric adjusted as conversations and discussions surfaced. See the table 3 for the full rubric used. The assessments were scored out of a total of 18.

Table 3: Rubric

Question	0pts.	1pt.	2pts.	Total pts.
1a	**Not Correct**	**Correct-Uses Other Method (i.e. long division)**	**Correct-Uses Primes (prime factorization) Identifying the use of prime factorization includes: (arrows pointing toward numbers in the equation, parenthesis around numbers in the equation-showing use of these numbers, multiplication with use of factors from the equation-i.e.: 3x5=15 (3,5, and 15)**	**/2**
1b	**Not Correct**	**Correct-Uses Other Method (i.e. long division)**	**Correct-Uses Primes (prime factorization)**	**/2**
1c	**Not Correct**	**Correct-Uses Other Method (i.e. long division)**	**Correct-Uses Primes (prime factorization)**	**/2**
1d	**Not Correct**	**Correct-Uses Other Method (i.e. long division)**	**Correct-Uses Primes (prime factorization)**	**/2**
			Section 1-Total pts.	**/8**
	0pts.		**1-8pts.**	**Total pts.**
2a	**No Factors**		**Total Factors Found: (Factors include factors used within work, i.e.: 3x5=15 (3,5, and 15) are all factors)**	**/8**
			Section 2A- Total pts.	**/8**
	0pts.	**1pt.**	**2pts.**	**Total pts.**
2b	**No written or visual representation of reasoning.**	**Shows written or visual representation of reasoning using other methods of reasoning. (long division)**	**Shows written or visual representation of reasoning using prime factorization.(prime factorization) Identifying the use of prime factorization includes: (arrows pointing toward numbers in the equation, parenthesis around numbers in the equation-showing use of these numbers, multiplication with use of factors from the equation-i.e.: 3x5=15 (3,5, and 15)**	
			Section 2B-Total pts.	**/2**

4. Results

The results were examined using a paired t-test with an alpha level set at .05. This study uses N=33 for analysis. Students were removed from the sample if they did not attend both days of the intervention, did not complete both the pre- and post-assessment, or completed less than 50% of one or both of the assessments.

Table 4: Total

	Pre-Assessment	Post-Assessment	Difference
Average	8.30303	12.06061	3.757576
Standard Deviation	4.111606	3.999526	4.437273

T(33)= 4.864615744; p=2.94419E-05

Table 4 shows the average and standard deviation for the pre-assessment, post-assessment and the difference between the two for the total scores on the assessments. The p-value is substantially less than the alpha level of .05. This means that the students had statistically significant gains in their ability to answer all of the questions on the test with a higher success rate after the intervention. Overall, this data shows the intervention is associated with an increase in ability to use prime factorization to find factors of numbers.

Table 5: Question 1

	Pre-Assessment	Post-Assessment	Difference
Average	3	5.363636	2.363636
Standard Deviation	1.436141	2.19115	2.329407

T(33)= 5.828976; p=1.78E-06

Table 5 shows the average and standard deviation for the pre-assessment, post-assessment, and the difference between the two for the score on question 1 of the assessments. Question 1 of the assessment tested the students' ability to distinguish between factors and non-factors of a number. Students were also expected to show the use of prime numbers in their explanation for why a number was a factor or not. The p-value for table 5 is much less than the alpha level of .05. This indicates that the intervention is associated with student gains in abilities to identify factors and non-factors of a number using the prime factorization of a number to help them with the identification process.

Table 6: Question 2a

	Pre-Assessment	Post-Assessment	Difference
Average	3.787879	5.272727	1.484848
Standard Deviation	2.654684	2.225881	2.751377

$T(33) = 3.100195$; $p=0.004015$

Table 6 shows the average and standard deviation for the pre-assessment, post-assessment, and the difference between the two for the score on question 2a of the assessments. Questions 2a tested the students' ability to identify factors of a number presented in prime factorization form. The p-value for table 6 still meets the alpha level p less than .05; this means that the intervention is associated with students being able to find more factors of a number after the intervention.

Table 7: Question 2b

	Pre-Assessment	Post-Assessment	Difference
Average	1.515152	1.424242	-0.09091
Standard Deviation	0.870388	0.867118	1.259058

$T(33) = -0.41478$; $p=0.68107$

Table 7 shows the average and standard deviation of the pre-assessment, post-assessment, and the difference between the two on question 2b. Questions 2b tests the students' ability to use the prime factors of a number to identify the factors. Students were assessed on their explanation abilities. As the data shows, students were actually less successful at explaining how they found the factors for the number. However, the p-value indicates that the intervention is not associated with any change in ability in this area.

5. Conclusion

The results of this study are very promising. The statistics show that students had great gains in all areas of the assessment, with the exception of explaining how they found the factors for the last question (i.e. question 2b). The results from the pre-assessment showed that students did not use the uniqueness of prime factorizations to help them in identifying factors and non-factors of a number presented in prime factor form. The results found show that students share a similar misunderstanding of identifying prime factors that PSTs have shown in previous research. Furthermore, the research conducted by Roscoe and Feldman showed improvement in PSTs understandings of this topic, which was replicated in this study with students in the 4^{th} and 5^{th} grades.[2] This seems to indicate that the instructional tasks used during the intervention helped to improve students' abilities in this area of mathematics. One limitation of this study is the lack of a control group for comparison of gains associated with the educational experience. A future study might better explain the impact of the intervention compared to traditional instruction in factor analysis where the FTA is not explored as a tool for finding factors.

The results point toward the need for more studies to develop the instructional tools to help students with their explanation abilities. The hope is for students to be able to successfully show and produce a written explanation for how prime factors of a number help to produce all of the factors for a number. It may be helpful to have student verbalize their understanding and then ask them to write it down after a successful discussion. However, the ultimate goal would be for students to be able to write out their understandings without needing to verbalize it first. Importantly, the Common Core State Standards in Mathematics (2010) have called for a focus on developing student's abilities to successfully reason through and explain difficult mathematical concepts. The mathematical practices for 4^{th} graders that this intervention most closely worked towards are: reason abstractly and quantitatively; construct viable arguments and critique the reasoning of others; look for and make sure of structure.[13] Students were challenged to construct their own knowledge and use the knowledge in new and unique situations. This challenge will hopefully lead to a richer and deeper understanding of factors and how numbers are uniquely made up of primes.

Secondly, it may be helpful and important to include some of Burkhart's ideas into this intervention. The use of physical blocks to build understanding of prime factorization and the creation of factors could help students develop a model for what is happening with the mathematics.[11] Students would be able to physically move around different pieces, primes, of the number to produce the factors. While the intervention, as is, was successful, it is always important for educators to think of more ideas to help students understand these important mathematical concepts and explain them.

This study produced unique and significant information about mathematical tasks for 4^{th} and 5^{th} grade students to develop their understandings of factors. While the study needs improvement in developing student's ability to explain their work, the results show that students were successful at developing a richer understanding of factors. The research points towards instructional methods that teachers can readily adopt to help them provide a deeper understanding of factors for their students.

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