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Rank of Recurrence Matrices

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Abstract

A recurrence relation is an equation that recursively defines a sequence of numbers, given one or more initial terms. An $m \times n$ recurrence matrix is a matrix whose entries read row-by-row are the terms of a sequence defined by a recurrence relation. The rank of a matrix is the maximum number of linearly independent columns or rows of the matrix. In 2014, Christopher Lee and Valerie Peterson proved that the maximum rank of a recurrence matrix is the order of the corresponding recurrence relation but that for order-two recurrence relations the rank drops if the ratio of the two initial terms of the sequence is an eigenvalue of the relation. Using the method of fundamental solutions, we generalize their result for order-two relations by showing that rank also drops if the n^{th} powers of the eigenvalues coincide. We then discuss more recent results by Sebastian Bozlee that determine the rank of recurrence matrices based on whether the relation can be written to have lower order and whether the n^{th} powers of the eigenvalues coincide in hopes of extending our results to recurrence relations of orders higher than two.

Keywords: Linear Algebra, Rank, Recurrence Relation, Matrix

1. Introduction

A recurrence relation is an equation that defines a sequence of numbers, given one or more initial terms. Real-world applications of recurrence relations include computer algorithms. The Fibonacci search technique, for example, is a computer algorithm that uses a divide-and-conquer method to locate the positions of certain Fibonacci numbers. These algorithms are also used in biology to predict, for example, the arrangement of branches on a tree, the flowering of an artichoke, and the uncurling of a fern [6]. Most importantly for our purposes, the formula for finding the sequential terms of the Fibonacci sequence is an example of a recurrence relation.

The terms defined by a recurrence relation can be put into a matrix, row-by-row, to yield a recurrence matrix. For example, a 3×3 recurrence matrix whose entries are the first nine numbers of the Fibonacci sequence $\{F_n\}$ defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ is

$$F = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 8 \\ 13 & 21 & 34 \end{bmatrix},$$

and like any matrix, it is possible to find its rank. Rank, which is defined as the maximum number of linearly independent rows or columns of the matrix (see, for instance, [4] or any other standard linear algebra textbook), is of great importance in solving systems of linear equations. In general, the number of equations in a system corresponds to the number of rows in that system's corresponding matrix. Three basic row operations (scaling all entries of a row by a nonzero constant, interchanging two rows, and replacing one row with the sum of itself and a multiple of another row) can be used to reduce all rows corresponding to redundant information in the system to zeros, and the number of remaining rows corresponding to essential information in the system is the rank of the matrix.

We will use these three row operations to find the rank of recurrence matrices whose entries come from sequences defined by homogeneous recurrence relations. Furthermore, we will compute formulas that will help us understand when a recurrence matrix has full rank, and when its rank drops. This has been shown in the order-two case in [5], but we prove their findings using the method of fundamental solutions [2]. With added care, we will also use the transpose of the recurrence matrices to help us understand when their ranks drop. The transpose of an $m \times n$ matrix R is defined as the $n \times m$ matrix denoted by R^T, whose columns are formed by using the corresponding rows of R (see, for instance, [4] or any standard linear algebra book). We are able to use R^T because $\text{rank}(R) = \text{rank}(R^T)$. Using R^T allows us to find formulas for row operations instead of finding linearly independent columns as in [5]. Through this study we hope to extend this method to matrices whose entries come from order-three homogeneous recurrence relations.

2. Arithmetic and Geometric Sequences

The simplest recurrence relation is the arithmetic sequence defined by the equation $a_k = a_1 + (k-1)x$ where a_1 is an initial seed, and x is a common difference added to each term to get the next term. By re-indexing if necessary, we assume that $a_1 \neq 0$, and we assume that $x \neq 0$ so that the sequence is not constant. Now, let an arithmetic matrix be a matrix whose entries come from an arithmetic sequence defined by an arithmetic recurrence relation.

The following proposition was made in [5] but was also observed in [3] and was the starting point for [7]. We will prove this proposition using our method rather than the method seen in [5].

Proposition 1. *Every $m \times n$ arithmetic matrix A with $m, n \geq 2$ has rank 2.*

Proof. As explained previously, we consider A^T whose entries read column-by-column are the terms of an arithmetic sequence $\{a_k\}$ defined by

$$a_k = a_1 + (k-1)x$$

with $a_1, x \neq 0$ as assumed above so that the (i, j)-entry of A^T is

$$a_{(j-1)n+i} = a_1 + ((j-1)n + i - 1)x.$$

Entries in the first row of A^T correspond to $i = 1$:

$$a_{(j-1)n+1} = a_1 + (j-1)nx$$

Likewise, entries in the second row of A^T correspond to $i = 2$:

$$a_{(j-1)n+2} = a_1 + ((j-1)n+1)x$$

For $i = 3, \ldots, n$, replacing Row i of A^T with

$$(\text{Row } i) + (i-2)(\text{Row } 1) - (i-1)(\text{Row } 2)$$

reduces entries in Row i to

$$a_1 + ((j-1)n + i - 1)x + (i-2)(a_1 + (j-1)nx) - (i-1)(a_1 + ((j-1)n+1))x) = 0.$$

Replacing Row 2 with

$$(\text{Row } 2) - \frac{a_1 + x}{a_1}(\text{Row } 1)$$

reduces entries in Row 2 to

$$a_1 + ((j-1)n+1)x - \frac{a_1 + x}{a_1}(a_1 + (j-1)nx) = -\frac{(j-1)nx^2}{a_1}$$

so the $(2,1)$-entry is 0 and all other entries in Row 2 are nonzero since $x \neq 0$. Thus, $\text{rank}(A) = \text{rank}(A^T) = 2$. □

Another example of a recurrence relation is the geometric sequence defined by the equation $a_k = ap^{k-1}$. Let a geometric matrix be a matrix whose entries come from a geometric sequence defined by a geometric recurrence relation. The following is also proven in [5], but we again prove it using our alternate method.

Proposition 2. *Every $m \times n$ geometric matrix G has rank 1.*

Proof. As before, we consider G^T whose entries read column-by-column are the terms of a geometric sequence $\{a_k\}$ defined by

$$a_k = ap^{k-1}$$

with the (i,j)-entry

$$a_{(j-1)n+i} = ap^{(j-1)n+i-1}.$$

Entries in the first row of G correspond to $i = 1$:

$$a_{(j-1)n+1} = ap^{(j-1)n}$$

For $i = 2, \ldots, n$, replacing Row i of G with

$$(\text{Row } i) - p^{i-1}(\text{Row } 1)$$

reduces entries in Row i to

$$ap^{(j-1)n+i-1} - p^{i-1}ap^{(j-1)n} = 0.$$

Hence, $\text{rank}(G) = \text{rank}(G^T) = 1$. □

3. Homogeneous Recurrence Relations

We now consider recurrence matrices whose entries are given by homogeneous recurrence relations. Let an order-r homogeneous recurrence relation be a relation whose k^{th} term is expressed as a linear combination of the preceding r terms. The following theorem was proven in [5], but yet again we prove it using our alternate method.

Theorem 1. *If R is an $m \times n$ matrix whose entries read row-by-row are given by an order-r homogeneous recurrence relation $a_k = \gamma_1 a_{k-1} + \cdots + \gamma_r a_{k-r}$, then rank$(R) \leq r$.*

Proof. As before, we consider R^T whose entries read column-by-column come from the order-r homogeneous recurrence relation

$$a_k = \gamma_1 a_{k-1} + \cdots + \gamma_r a_{k-r}$$

so the (i, j)-entry of R^T is

$$a_{(j-1)n+i} = \gamma_1 a_{(j-1)n+i-1} + \cdots + \gamma_r a_{(j-1)n+i-r}.$$

For $i = n, ..., (r + 1)$, replacing Row i of R with

$$(\text{Row } i) - \gamma_1(\text{Row } (i-1)) - \cdots - \gamma_r(\text{Row } (i-r))$$

reduces entries in Row i to

$$a_{(j-1)n+i} - \gamma_1 a_{(j-1)n+i-1} - \cdots - \gamma_r a_{(j-1)n+i-r} = 0.$$

Hence, rank(R) = rank$(R^T) \leq r$. □

The following two examples show the two instances in which the rank of a matrix can drop. These instances will be further investigated in this paper.

Example 3.1. Consider the order-two recurrence relation $a_k = 3a_{k-1} - 2a_{k-2}$ with initial seed $a_0 = 1$ and $a_1 = 2$. The corresponding recurrence matrix is

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 16 & 32 \\ 64 & 128 & 256 \end{bmatrix}.$$

Given that each row is a multiple of the first row, we can reduce rows beyond the first to zero; therefore, this matrix has rank 1 despite it coming from a order-two recurrence relation. This can be explained by the fact that it can also be written as $a_k = 2^k$ satisfying the recurrence relation $a_k = 2a_{k-1}$, which is an order-one recurrence relation, so the rank is bounded above by 1 rather than 2. This happens because the recurrence relation can be written to have a lower order.

Example 3.2. Let's consider the recurrence relation $a_k = a_{k-2}$ with initial seeds $a_0 = 4$ and $a_1 = 0$. This means that $a_k = 4$ for even k, and $a_k = 0$ for odd k. If we construct a 3×3 matrix from this sequence,

$$R = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

we have a matrix with rank 2; however, if we construct a 4×4 matrix from the sequence,

$$R = \begin{bmatrix} 4 & 0 & 4 & 0 \\ 4 & 0 & 4 & 0 \\ 4 & 0 & 4 & 0 \\ 4 & 0 & 4 & 0 \end{bmatrix}$$

we have a matrix that has rank 1 because all rows are a multiple of the first row. This happens due to the width of the matrix. This situation will be further analyzed in the theorems to come.

4. Eigenvalues and Fundamental Solutions

Eigenvalues of an order-two homogeneous recurrence relation are obtained by solving the quadratic equation corresponding to the recurrence relation. We now show how the eigenvalues and fundamental solutions for an order-two homogeneous recurrence relation are found.

Suppose we are given an order-two homogeneous recurrence relation

$$a_k = \gamma_1 a_{k-1} + \gamma_2 a_{k-2}.$$

If we let $a_k = x^2$, $a_{k-1} = x$, and $a_{k-2} = 1$, we obtain the equation

$$x^2 - \gamma_1 x - \gamma_2 = 0.$$

Solving this quadratic equation gives the eigenvalues of the equation. If the eigenvalues λ_1 and λ_2 are distinct, then the fundamental solutions are λ_1^{k-1} and λ_2^{k-1} and the general solution of the recurrence relation is

$$a_k = a\lambda_1^{k-1} + b\lambda_2^{k-1}.$$

If the eigenvalue λ is repeated, then the fundamental solutions are λ^{k-1} and $k\lambda^{k-1}$ and the general solution of the recurrence relation is

$$a_k = a\lambda^{k-1} + bk\lambda^{k-1}.$$

We use this method of finding the general solutions of a recurrence relation to prove the following generalization of a theorem from [5].

Theorem 2. *An $m \times n$ recurrence matrix R whose entries come from an order-two homogeneous recurrence relation has rank 1 if and only if $\frac{a_2}{a_1}$ is an eigenvalue of the relation with initial terms a_1 and a_2 or the relation has distinct eigenvalues whose n^{th} powers coincide.*

Proof. As before, we consider R^T whose entries read column-by-column come from an order-two homogeneous recurrence relation

$$a_k = \gamma_1 a_{k-1} + \gamma_2 a_{k-2}$$

so the (i,j)-entry of R^T is

$$a_{(j-1)n+i} = \gamma_1 a_{(j-1)n+i-1} + \gamma_2 a_{(j-1)n+i-2}.$$

As before, for $i = n, ..., 3$, replacing Row i of R^T with

$$(\text{Row } i) - \gamma_1(\text{Row } (i-1)) - \gamma_2(\text{Row } (i-2))$$

reduces entries in Row i to

$$a_{(j-1)n+i} - \gamma_1 a_{(j-1)n+i-1} - \gamma_2 a_{(j-1)n+i-2} = 0.$$

Thus, $\text{rank}(R^T) \leq 2$, and we determine when $\text{rank}(R^T) = 1$ using the method of fundamental solutions for the following two cases.

4.1 Case 1: One repeated eigenvalue

If $x^2 - \gamma_1 x - \gamma_2 = 0$ has a single repeated eigenvalue λ, then its fundamental solutions are λ^{k-1} and $k\lambda^{k-1}$ and its general solution is $a_k = a\lambda^{k-1} + bk\lambda^{k-1}$. To find a and b, we solve the system of equations $a_1 = a + b, a_2 = a\lambda + 2b\lambda$ by row reducing the augmented matrix

$$\begin{bmatrix} 1 & 1 & a_1 \\ \lambda & 2\lambda & a_2 \end{bmatrix}$$

which yields $a = \frac{2\lambda a_1 - a_2}{\lambda}$ and $b = \frac{a_2 - \lambda a_1}{\lambda}$, so the general solution is

$$a_k = \lambda^{k-1}(2-k)a_1 + \lambda^{k-2}(k-1)a_2$$

and the (i,j)-entry of R^T (read column-by-column) is

$$a_{(j-1)n+i} = \lambda^{(j-1)n+i-1}(2-(j-1)n-i)a_1 + \lambda^{(j-1)n+i-2}((j-1)n+i-1)a_2.$$

Entries in the first row of R^T correspond to $i = 1$:

$$a_{(j-1)n+1} = \lambda^{(j-1)n}(1-(j-1)n)a_1 + \lambda^{(j-1)n-1}(j-1)na_2$$

Entries in the second row of R^T correspond to $i = 2$:

$$a_{(j-1)n+2} = \lambda^{(j-1)n+1}(1-j)na_1 + \lambda^{(j-1)n}((j-1)n+1)a_2$$

Replacing Row 2 with (Row 2) $- \frac{a_2}{a_1}$(Row 1) reduces entries in Row 2 to

$$-\frac{\lambda^{(j-1)n-1}(j-1)n}{a_1}(\lambda a_1 - a_2)^2$$

which is 0, assuming that $\lambda \neq 0$, if and only if $\lambda = \frac{a_2}{a_1}$. Hence, $\text{rank}(R) = \text{rank}(R^T) = 1$ if and only if $\lambda = \frac{a_2}{a_1}$.

4.2 Case 2: Two distinct eigenvalues

If $x^2 - \gamma_1 x - \gamma_2 = 0$ has two distinct eigenvalues, λ_1 and λ_2, then its fundamental solutions are λ_1^{k-1} and λ_2^{k-1}, and its general solution is $a_k = a\lambda_1^{k-1} + b\lambda_2^{k-1}$. To find a and b, we solve the system of equations $a_1 = a + b$, $a_2 = a\lambda_1 + b\lambda_2$ by row reducing the augmented matrix

$$\begin{bmatrix} 1 & 1 & a_1 \\ \lambda_1 & \lambda_2 & a_2 \end{bmatrix}.$$

Row reduction yields $a = \frac{\lambda_2 a_1 - a_2}{\lambda_2 - \lambda_1}$ and $b = \frac{a_2 - \lambda_1 a_1}{\lambda_2 - \lambda_1}$, so the general solution is

$$a_k = \frac{\lambda_2 \lambda_1^{k-1} - \lambda_1 \lambda_2^{k-1}}{\lambda_2 - \lambda_1} a_1 + \frac{\lambda_2^{k-1} - \lambda_1^{k-1}}{\lambda_2 - \lambda_1} a_2$$

and the (i, j)-entry of R^T (read column-by-column) is

$$a_{(j-1)n+i} = \frac{\lambda_2 \lambda_1^{(j-1)n+i-1} - \lambda_1 \lambda_2^{(j-1)n+i-1}}{\lambda_2 - \lambda_1} a_1 + \frac{\lambda_2^{(j-1)n+i-1} - \lambda_1^{(j-1)n+i-1}}{\lambda_2 - \lambda_1} a_2.$$

Entries in the first row of R^T correspond to $i = 1$:

$$a_{(j-1)n+1} = \frac{\lambda_2 \lambda_1^{(j-1)n} - \lambda_1 \lambda_2^{(j-1)n}}{\lambda_2 - \lambda_1} a_1 + \frac{\lambda_2^{(j-1)n} - \lambda_1^{(j-1)n}}{\lambda_2 - \lambda_1} a_2$$

Entries in the second row of R^T correspond to $i = 2$:

$$a_{(j-1)n+2} = \frac{\lambda_2 \lambda_1^{(j-1)n+1} - \lambda_1 \lambda_2^{(j-1)n+1}}{\lambda_2 - \lambda_1} a_1 + \frac{\lambda_2^{(j-1)n+1} - \lambda_1^{(j-1)n+1}}{\lambda_2 - \lambda_1} a_2$$

Replacing Row 2 with (Row 2) $- \frac{a_2}{a_1}$(Row 1) reduces entries in Row 2 to

$$\frac{\lambda_1^{(j-1)n} - \lambda_2^{(j-1)n}}{(\lambda_2 - \lambda_1) a_1} (a_2 - \lambda_1 a_1)(a_2 - \lambda_2 a_1)$$

which is 0 if and only if $\lambda_1 = \frac{a_2}{a_1}$, $\lambda_2 = \frac{a_2}{a_1}$, or $\lambda_1^n = \lambda_2^n$, so $\text{rank}(R) = \text{rank}(R^T) = 1$ iff $\lambda_1 = \frac{a_2}{a_1}$, $\lambda_2 = \frac{a_2}{a_1}$, or $\lambda_1^n = \lambda_2^n$. □

4.3 Characterizing Rank Drops

Rank drops that take place when the n^{th} powers of the eigenvalues coincide were not observed in [5] but were later characterized along with rank drops that take place when a recurrence relation can be written to have a lower order (i.e., a minimal order recurrence) in [1]. In particular, the following theorem from [1] finds the minimal order recurrence relation satisfied by a recurrence relation.

Theorem 3. *Let $\{a_k\}$ be a sequence given by an order-r recurrence relation with q distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_q$ with respective multiplicities $k_1, k_2, \ldots, k_q$. Let $K_{l,i}$ be the unique constants so that*

$$a_k = \sum_{l=1}^{q} \sum_{i=1}^{k_l} K_{l,i} (k^{i-1} \lambda_l^k)$$

Let M_l be the maximal value of i so that $K_{l,i}$ is nonzero, or zero if $K_{l,i}$ is zero for all i. Then the minimal order recurrence satisfied by $\{a_k\}$ is the recurrence with the characteristic polynomial $f(\lambda) = \prod_{l=1}^{q} (\lambda - \lambda_l)^{M_l}$.

This next theorem from [1] characterizes rank drops that take place when the n^{th} powers of the eigenvalues coincide; such rank drops are called "width drops."

Theorem 4. *Let $\{a_k\}$ be an order-r recurrence sequence with q distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_q$ with multiplicities $k_1, k_2, \ldots, k_q$ respectively. Let S_n be the set of distinct values taken by $\lambda_1^n, \ldots, \lambda_q^n$. Then if $m, n \geq r$,*

$$\text{rank } M_{m,n}(a_k) = \sum_{s \in S_n} \max_l \{k_l : \lambda_l^n = s\}$$

where $M_{m,n}(a_k)$ is the $m \times n$ recurrence matrix M whose entries are the terms from the recurrence sequence $\{a_k\}$.

Finally, this last theorem from [1] finds all possible rank drops for order-two homogeneous recurrence relations.

Theorem 5. *If R is an $m \times n$ matrix with $m, n \geq 2$ whose entries (read row-by-row) are given by an order-two homogeneous recurrence relation $a_k = \gamma_1 a_{k-1} + \gamma_2 a_{k-2}$, then*

$$\text{rank}(R) = \begin{cases} 0 & \textit{if } a_0 = a_1 = 0 \\ 1 & \textit{if } a_1^2 - \gamma_1 a_1 a_0 - \gamma_2 a_0^2 = 0 \\ 1 & \textit{if } \gamma_1^2 + 4\gamma_2 \neq 0 \textit{ and } \left(\frac{\gamma_1 + \sqrt{\gamma_1^2 + 4\gamma_2}}{\gamma_1 - \sqrt{\gamma_1^2 + 4\gamma_2}}\right)^n = 1 \\ 2 & \textit{else} \end{cases}$$

These are the same results we found in Theorem 2. We now use the approach of [1] for the order-three homogeneous recurrence relation case. Note that there are several differences from our original approach. In [1], the sequence $\{a_k\}$ begins at $k = 0$ whereas we began $\{a_k\}$ at $k = 1$. Moreover, instead of using row operations to determine when rank drops, [1] uses Cramer's Rule. This particular method calls for an $n \times n$ invertible matrix A such that the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries of the form $x_i = \frac{\det A_i(\mathbf{b})}{\det A}$ where $i = 1, \ldots, n$. In particular, the adjoint of the matrix A, denoted $\text{adj}(A)$, divided by the determinant of A, denoted $\det(A)$, help form the solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$.

5. Order-three homogeneous recurrence relations

Suppose we are given an order-three homogeneous recurrence relation

$$a_k = \gamma_1 a_{k-1} + \gamma_2 a_{k-2} + \gamma_3 a_{k-3}.$$

If we let $a_k = x^3$, $a_{k-1} = x^2$, $a_{k-2} = x$, and $a_{k-3} = 1$, we obtain the equation

$$x^3 - \gamma_1 x^2 - \gamma_2 x - \gamma_3 = 0.$$

Solving this cubic equation gives the eigenvalues of the equation. If the eigenvalues λ_1, λ_2, and λ_3 are distinct, then the fundamental solutions are λ_1^{k-1}, λ_2^{k-1}, and λ_3^{k-1}, and the general solution of the recurrence relation is

$$a_k = a\lambda_1^{k-1} + b\lambda_2^{k-1} + c\lambda_3^{k-1}.$$

If the eigenvalue λ is repeated with multiplicity 3, then the fundamental solutions are λ^{k-1}, $k\lambda^{k-1}$, and $k^2\lambda^{k-1}$, and the general solution of the recurrence relation is

$$a_k = a\lambda^{k-1} + bk\lambda^{k-1} + ck^2\lambda^{k-1}.$$

If one eigenvalue λ_1 is repeated with multiplicity 2, and the other, λ_2, is distinct, then the fundamental solutions are λ_1^{k-1}, $k\lambda_1^{k-1}$, and λ_2^{k-1}, and the general solution of the recurrence relation is

$$a_k = a\lambda_1^{k-1} + bk\lambda_1^{k-1} + c\lambda_2^{k-1}.$$

We will use the method of finding fundamental solutions as in the order-two case to determine when an order-three homogeneous recurrence relation yields a matrix that has maximum rank of 3, and when it drops to 2 or even 1.

Theorem 6. *An $m \times n$ recurrence matrix R whose entries come from an order-three homogeneous recurrence relation with seeds a_0, a_1, a_2 has rank 2 if and only if $a_2 = (\lambda_1 + \lambda_2)a_1 - \lambda_1\lambda_2 a_0$ where λ_1 and λ_2 are (not necessarily distinct) eigenvalues of the recurrence relation or if $\lambda_1^n = \lambda_2^n$ for two distinct eigenvalues of the recurrence relation, and rank 1 if and only if, in addition, $a_1 = \lambda a_0$ where λ is an eigenvalue of the recurrence relation, or $\lambda_1^n = \lambda_2^n = \lambda_3^n$ for three distinct eigenvalues of the recurrence relation.*

Proof. Entries of R^T (read column-by-column) come from an order-three homogeneous recurrence relation

$$a_k = \gamma_1 a_{k-1} + \gamma_2 a_{k-2} + \gamma_3 a_{k-3}$$

so the (i, j)-entry of R is

$$a_{(j-1)m+i} = \gamma_1 a_{(j-1)m+i-1} + \gamma_2 a_{(j-1)m+i-2} + \gamma_3 a_{(j-1)m+i-3}.$$

As before, for $i = m, ..., 4$, replacing Row i of R with

$$(\text{Row } i) - \gamma_1(\text{Row } (i-1)) - \gamma_2(\text{Row } (i-2)) - \gamma_3(\text{Row } (i-3))$$

reduces entries in Row i to

$$a_{(j-1)m+i} - \gamma_1 a_{(j-1)m+i-1} - \gamma_2 a_{(j-1)m+i-2} - \gamma_3 a_{(j-1)m+i-2} = 0.$$

Thus, $\text{rank}(R) \leq 3$, and we determine when $\text{rank}(R) = 2$ or $\text{rank}(R) = 1$ using the method of fundamental solutions.

5.1 Case 1: One repeated eigenvalue

To find a, b, and c in $a_k = a\lambda^k + bk\lambda^k + ck^2\lambda^k$ we use Cramer's rule on the system

$$\begin{bmatrix} 1 & 0 & 0 \\ \lambda & \lambda & \lambda \\ \lambda^2 & \lambda^2 & \lambda^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix},$$

which after left multiplying by the inverse of the coefficient matrix on both sides, and dividing by the determinant of the coefficient matrix, yields the solution

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2\lambda^3} \begin{bmatrix} 2\lambda^3 a_0 \\ -\lambda(3a_0\lambda^2 - 4a_1\lambda + a_2) \\ \lambda(a_0\lambda^2 - 2a_1\lambda + a_2) \end{bmatrix}.$$

So if R is the corresponding recurrence matrix R, then

$$\operatorname{rank}(R) = \operatorname{rank}\,(R^T) = \begin{cases} 3 & \text{if } a,b,c \neq 0 \\ 2 & \text{if } a,b \neq 0, \text{ but } c = 0, \text{ i.e., } a_2 = 2\lambda a_1 - \lambda^2 a_0 \\ 1 & \text{if } a \neq 0, \text{ but } b = c = 0, \text{ i.e., } a_1 = \lambda a_0 \text{ and } a_2 = \lambda^2 a_0 \\ 0 & \text{if } a = b = c = 0 \end{cases}$$

By Theorem 4, $\{s_n\}$ is the set of distinct values taken by the roots of its corresponding characteristic polynomial. In this case, $s_n = \{\lambda_1^n\}$ and rank $R_{m,n}(a_k) = \max_l\{K_l : \lambda_1^n = \lambda_1^n\} = 3$. Thus, there is no possible width drop in this particular case.

5.2 Case 2: Three distinct eigenvalues

To find a, b, and c in $a_k = a\lambda_1^k + k\lambda_2^k + c\lambda_3^k$ we use Cramer's rule on the system

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix},$$

which after left multiplying by the inverse of the coefficient matrix on both sides, and dividing by the determinant of the coefficient matrix, yields the solution

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{a_0\lambda_1\lambda_2 - a_1\lambda_1 - a_1\lambda_2 + a_2}{\lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_1\lambda_2 - \lambda_3^2} \\ \frac{-a_0\lambda_1\lambda_3 + a_1\lambda_1 + a_1\lambda_3 - a_2}{\lambda_1\lambda_2 - \lambda_1\lambda_3 + \lambda_1\lambda_2 - \lambda_2^2} \\ \frac{a_0\lambda_2\lambda_3 - a_1\lambda_2 - a_1\lambda_3 + a_2}{\lambda_2\lambda_3 - \lambda_1\lambda_3\lambda_1\lambda_2 + \lambda_1^2} \end{bmatrix}.$$

So if R is the corresponding recurrence matrix R, then

$$\operatorname{rank}(R) = \operatorname{rank}\,(R^T) = \begin{cases} 3 & \text{if } a,b,c \neq 0 \text{ and } \lambda_1^n \neq \lambda_2^n,\ \lambda_1^n \neq \lambda_3^n, \text{ or } \lambda_2^n \neq \lambda_3^n \\ 2 & \text{if } a,b \neq 0, \text{ but } c = 0, \text{ i.e., } a_2 = (\lambda_2 + \lambda_3)a_1 - \lambda_2\lambda_3 a_0 \\ 2 & \text{if } \lambda_1^n = \lambda_2^n \text{ or } \lambda_1^n = \lambda_3^n \text{ or } \lambda_2^n = \lambda_3^n \\ 1 & \text{if } a \neq 0, \text{ but } b = c = 0, i.e., a_1 = \lambda_1 a_0 \\ 1 & \text{if } \lambda_1^n = \lambda_2^n = \lambda_3^n \\ 0 & \text{if } a = b = c = 0 \end{cases}$$

By Theorem 4, $s_n = \{\lambda_1^n, \lambda_2^n, \lambda_3^n\}$ is the set of distinct values taken on by the roots of its corresponding characteristic polynomial, and rank $R_{m,n}(a_k) = \max_l\{K_l : \lambda_1^n = \lambda_1^n\} + \max_l\{K_l : \lambda_1^n = \lambda_2^n\} + \max_l\{K_l : \lambda_1^n = \lambda_3^n\} = 3$. Thus, there exists a width rank drop to 2 when $\lambda_1^n = \lambda_2^n$, $\lambda_1^n = \lambda_3^n$, or $\lambda_2^n = \lambda_3^n$, and a width rank drop to 1 if $\lambda_1^n = \lambda_2^n = \lambda_3^n$.

5.3 Case 3: One repeated and one distinct eigenvalue

To find a, b, and c in $a_k = a\lambda_1^k + b\lambda_2^k + ck\lambda_2^k$ we use Cramer's rule on the system

$$\begin{bmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_1^2 & \lambda_2^2 & 2\lambda_2^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

which after left multiplying by the inverse of the coefficient matrix on both sides, and dividing by the determinant of the coefficient matrix, yields the solution

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{\lambda_2^2 a_0 - 2\lambda_2 a_1 + a_2}{(\lambda_1 - \lambda_2)^2} \\ \frac{\lambda_1^2 a_0 - 2\lambda_1\lambda_2 a_0 + 2\lambda_2 a_1 - a_2}{(\lambda_1 - \lambda_2)^2} \\ \frac{\lambda_1 a_1 - \lambda_1\lambda_2 a_0 + \lambda_2 a_1 - a_2}{\lambda_2(\lambda_1 - \lambda_2)} \end{bmatrix}$$

So if R is the corresponding recurrence matrix R, then

$$\text{rank}(R) = \text{rank } (R^T) = \begin{cases} 3 & \text{if } a, b, c \neq 0 \text{ and } \lambda_1^n \neq \lambda_2^n \\ 2 & \text{if } \lambda_1^n = \lambda_2^n \text{ or } a, b \neq 0, \text{ but } c = 0, \text{ i.e., } a_2 = (\lambda_1 + \lambda_2)a_1 - \lambda_1\lambda_2 a_0 \\ 1 & \text{if } a \neq 0, \text{ but } b = c = 0, \text{ i.e., } \lambda_1 = \frac{a_1}{a_0} \\ 0 & \text{if } a = b = c = 0 \end{cases}$$

By Theorem 4, $s_n = \{\lambda_1^n, \lambda_2^n\}$ is the set of distinct values taken on by the roots of its corresponding characteristic polynomial, and rank $R_{m,n}(a_k) = \max_l\{K_l : \lambda_1^n = \lambda_1^n\} + \max_l\{K_l : \lambda_1^n = \lambda_2^n\} = 3$. Thus, there exists a width rank drop to 2 when $\lambda_1^n = \lambda_2^n$, and a width rank drop to 1 is non-existent since there is not another possible pair of eigenvalues' n^{th} power that can coincide. □

In general, the rank of an $m \times n$ recurrence matrix will drop if we can rewrite its corresponding recurrence relation to have lower order or when the distinct eigenvalues' n^{th} powers coincide (i.e., a width drop). See Examples 3.1 and 3.2 for an example of each (note that the eigenvalues in Example 3.2 are ± 1, so the rank drops only when the matrix has even width, i.e., an even number of columns).

6. Further Research

We will begin the generalization for order-four homogeneous recurrence relations using Bozlee's method to determine when the relations can be rewritten to have order 3, 2, or 1 and when the rank of the corresponding recurrence matrices drop.

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