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On the Relativistic Motion of Projectiles: On Range and Vertical Height

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Abstract

The properties of projectiles approaching the speed of light were studied mathematically and graphically, and compared to those of nonrelativistic projectiles. We found that the angle at which the range is equal to the vertical height for relativistic projectiles depends on speed, and ranges from 76° to 80.3°. This differs from nonrelativistic ones, whose angle is independent of speed and constant at 76 °. Different scenarios were considered using protons and electrons in vertical force fields to obtain our results.

Keywords: Projectile Motion, Special Relativity, SUNY Potsdam

1. Introduction

Projectiles are defined as objects moving through space under the influence of a constant downward force (such as force of gravity).[1] Examples of normal projectile motion includes the motion of baseballs, bullets, and water fountains. The kinematics of these normal, nonrelativistic projectiles are well understood and documented in virtually every fundamental physics textbook.[2] Some critical properties of nonrelativistic projectiles are listed below.[1-5] All x-axis and y-axis components are subscripted and refer to horizontal and vertical components respectively.

- $v_{xo} = v_o \cos\theta$ and $v_{yo} = v_o \sin\theta$. Here, v_o is initial velocity and θ is the angle of projection.
- $F = mg$. Here, F is vertical downward force, m is mass, and g is acceleration due to gravity.
- Motion along the x-direction and y-direction are independent.
- Horizontal acceleration, $a_x = 0$, since there is no force along x
- Horizontal velocity, $v_x = v_{xo}$ and horizontal displacement, $x = x_o + v_{xo}t$. Here, t is time.
- Vertical acceleration, $a_y = -g$.
- Vertical velocity, $v_y = v_{yo} - gt$
- Vertical displacement, $y = y_o + v_{yo}t - ½\, gt^2$
- $Time\ of\ flight\ (T) = 2v_o \sin\theta/g$
- The path is parabolic.
- Range, $R = v_0^2 \sin 2\theta/g$

- Maximum height, $H = v_0^2 sin^2\theta/g$
- Range is a maximum at 45°

Numerous properties of objects approaching the speed of light have been documented, such as time dilation, length contraction, relativistic momentum, and an increase in mass.[6] These are only observable or meaningful when an object, such as subatomic particles in atoms, is moving at speeds comparable to the speed of light.

2. Methodology

Analysis of relativistic projectile motion was done using MATLAB©, Microsoft Excel©, and Origin Pro 8©. For different masses, a proton and electron are used as test particles at varying speeds in a uniform downward electric force of 10^{-15} N in a reference frame of a stationary observer. It is assumed there is no magnetic force acting upon the particle.

For a non-relativistic projectile, the range (*R*) and vertical height (*H*) are given by the two equations:

$$R = \frac{v_o^2 sin2\theta}{g} \tag{1}$$

$$H = \frac{v_o^2 sin^2\theta}{2g} \tag{2}$$

Setting these two distance equations equal and solving for θ yields a projection angle of 76°, at which range and maximum vertical height are equal for nonrelativistic projectiles.

The analysis for relativistic projectiles was more complex[7-8]. Newton's second law of motion gives:

$$F = \frac{dp}{dt} \tag{3}$$

Here, *dp/dt* is the rate of change in relativistic momentum. Relativistic momentum is given by the equation:

$$P \ = \ mv\gamma \tag{4}$$

Here, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, *m* is the rest mass of the projectile, *v* is its speed, and *c* is the speed of light.[8] Since force in the *x*-direction is zero, whereas the force in the *y*-direction is $-F$, equation (3) shows that:

$$\frac{dp_x}{dt} = 0 \tag{5}$$

$$\frac{dp_y}{dt} = -F \tag{6}$$

For a projectile of initial momentum p_o projected at an angle θ, the x and y-components equations of momentum turn into:

$$\gamma m v_x = p_o cos\theta \tag{7}$$

$$\gamma m v_y = p_o sin\theta - Ft \tag{8}$$

Rearranging these equations for the x and y components of velocity results in the following:

$$v_x(t) = \frac{cp_o cos\theta}{\sqrt{p_o^2+(Ft)^2+(mc)^2-2p_oFtsin\theta}} \tag{9}$$

$$v_y(t) = \frac{-Ftc+ cp_o cos\theta}{\sqrt{p_o^2+(Ft)^2+(mc)^2-2p_oFtsin\theta}} \tag{10}$$

These equations were integrated to obtain the distances traveled by the projectile in the *x* and *y* directions[8]:

$$x(t) = \frac{cp_o cos\theta}{F} ln\left[\frac{Ft+\sqrt{\frac{E_o^2}{c^2}+(Ft)^2-2p_oFtsin\theta}-p_o sin\theta}{\frac{E_o}{c}-p_o sin\theta}\right] \tag{11}$$

$$y(t) = \frac{c}{F}\left[\frac{E_o}{c} - \sqrt{\frac{E_o^2}{c^2} + (Ft)^2 - 2p_oFtsin\theta}\right] \tag{12}$$

Here, E_0 is the initial energy of the projectile, given by:

$$E_0{}^2 = p_0{}^2c^2 + m^2c^4 \tag{13}$$

The above equations (11) and (12) were then plotted with the masses of a proton and electron at .1c and .99c at various projection angles. This is shown in (Figure 1) and (Figure 2).

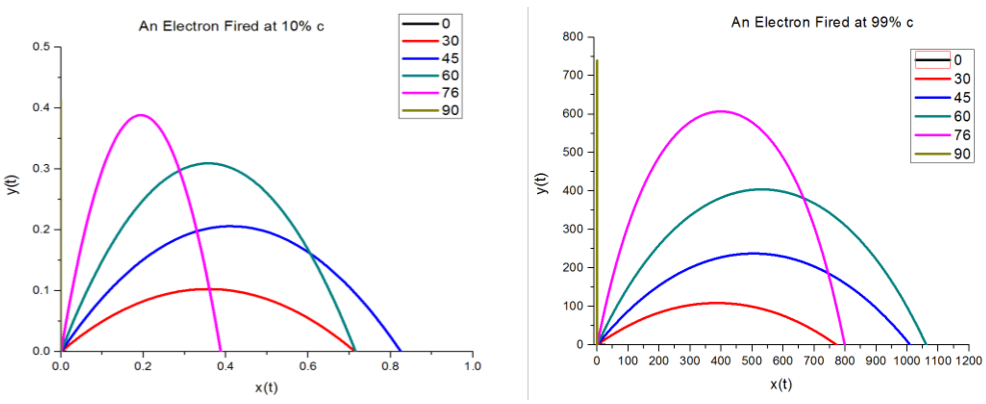


Figure 1. The projection profile of an electron fired at various angles at 0.1*c* (A) and 0.99*c* (B). As shown, the range (x) is equal to the maximum height (y) at approximately 76° at 0.1*c*, but not at 0.99*c*. Note that the scale had to be changed in side B, as the electron was propelled much further at nearly 10 times the speed on the left.

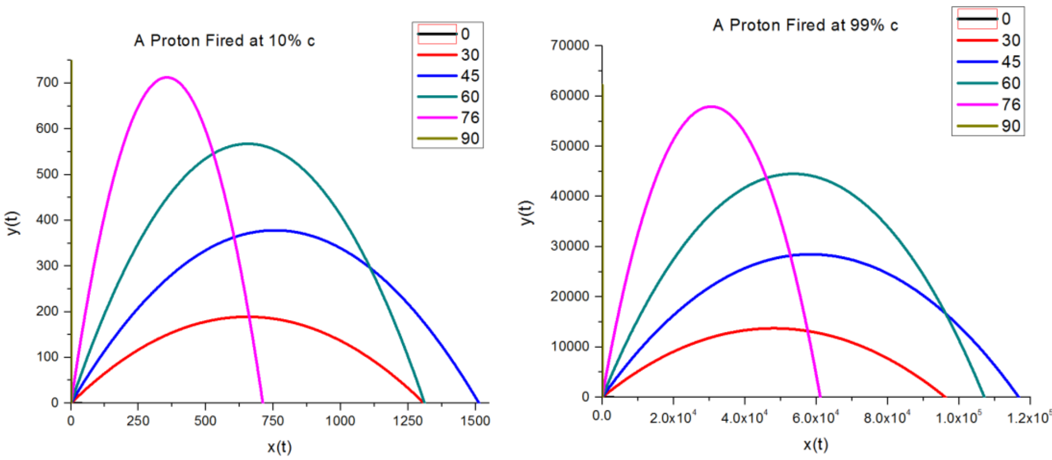


Figure 2. The projection profile of a proton fired at various angles at 0.1*c* (A) and 0.99*c* (B). Similarly to Figure 1, the range (x) is equal to the maximum height (y) at approximately 76° at 0.1*c*, but not at 0.99*c*. Note that the scale had to be changed in both cases, as the more massive proton travels farther than the less massive electron.

In order to get a better understanding of how the particles are acting as the speed increases, more algebraic analysis and graphical analysis was done.

Time of flight, *T*, is found by setting the vertical position equation (12) equal to zero. The result is:

$$T = \frac{2p_o sin\theta}{F} \tag{14}$$

Since a projectile has zero vertical motion at its zenith, maximum vertical height is found by setting the vertical velocity equation (10) equal to zero. This gives the time to reach maximum height as

$$t = \frac{p_o sin\theta}{F} \tag{15}$$

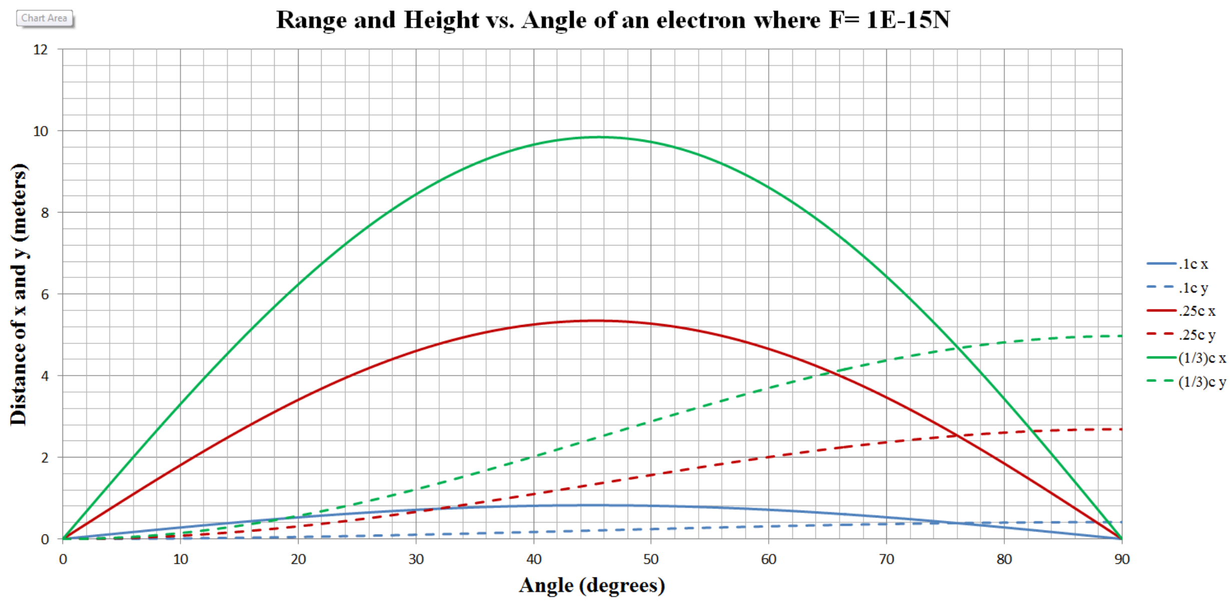
Finally, equation (14) is substituted for *t* in equation (11) and equation (15) is substituted for *t* in equation (12). The results are equations for range and maximum height for projectiles:

$$R(\theta) = \frac{cp_o cos\theta}{F} ln\left[\frac{p_o sin\theta + \frac{E_o}{c}}{\frac{E_o}{c} - p_o sin\theta}\right] \tag{16}$$

$$H(\theta) = \frac{c}{F}\left[\frac{E_o}{c} - \sqrt{\frac{E_o^2}{c^2} - p_o^2 sin^2\theta}\right] \tag{17}$$

3. Graphical Analysis

The scope of the study is to find the angle for which the range of a relativistic projectile is the same as its vertical height. To find the angle θ at which the range and maximum height are equal, we set the preceding equations (16) and (17) equal to each other and solved graphically for projection angle θ. Treating $R(\theta)$ and $H(\theta)$ as two independent functions of θ, they were graphed on the same plot, using known masses of a proton and electron, $F = 10^{-15}$ N and speeds of 0.1*c*, 0.25*c*, and 0.3*c*; their point of intersection represents the coordinates (θ, Range) at which the two are equal. This plot is shown and analyzed in (Figure 3).



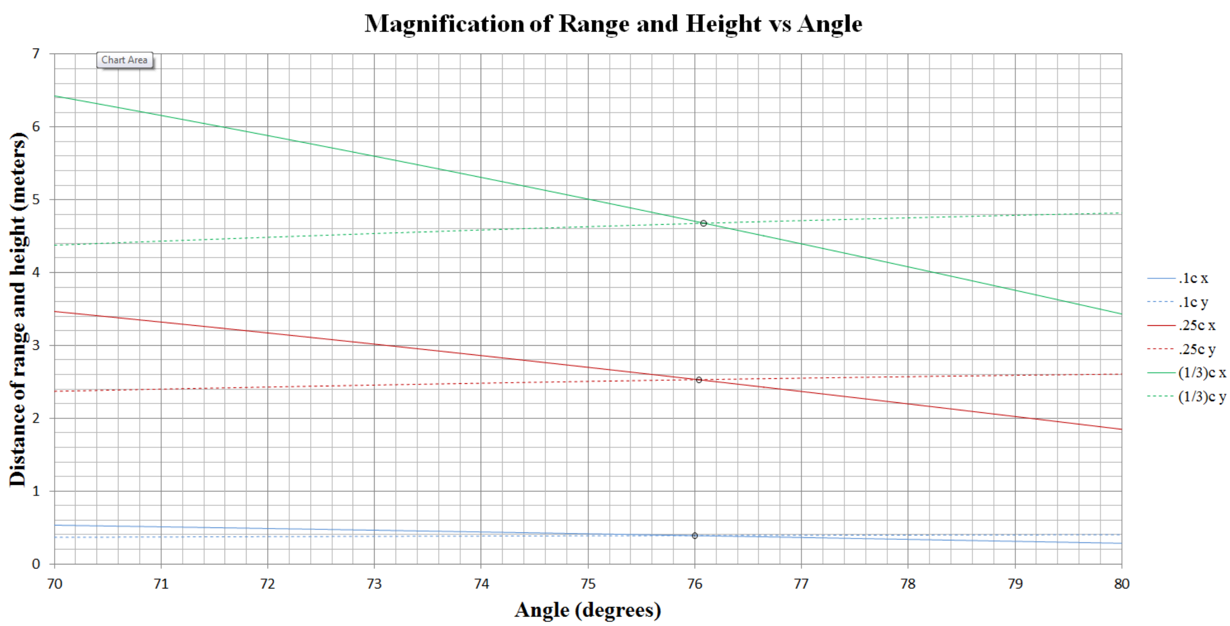


Figure 3. The plot of range (solid lines) and height (dashed lines) as a function of θ for 0.1*c*, 0.25*c*, and 0.3*c* is shown on the top graph. Here the intersecting lines represent the angle and velocity for which range is equal to height. The bottom graph is an enlargement of the top graph to help visualize the effect of this relatively small increase in speed. It can be clearly seen that as the velocity of the projectile increases, the angle at which range equals height increases.

The intersection of the range and height curves occurs at nearly exactly 76° for 0.1*c*, however this angle shifts up for speeds of 0.25*c* and 0.3*c*. To get a more accurate representation of the particles' behavior with increasing speed, MATLAB© was used to make two very large matrices of *R* and *H* for increasing increments of *v* and θ. Each column (increasing angle) of the *R* matrix was compared to the corresponding column of *H*. Each value in the columns of *R* was subtracted from each value in the columns of *H* individually. The resulting difference matrix then gave the smallest value of *H*-*R* for increasing increments of *v*. The smallest value in these columns will give angle and speed at which range is close to the maximum height. The resulting angle and speed was then exported to Microsoft Excel© for graphical interpretation. Projection angle was determined graphically for speeds from 0.01c to 0.99c, with increments of 0.01*c*. The resulting graphs in Figure 4 were an increase that asymptotically approached the speed of light.

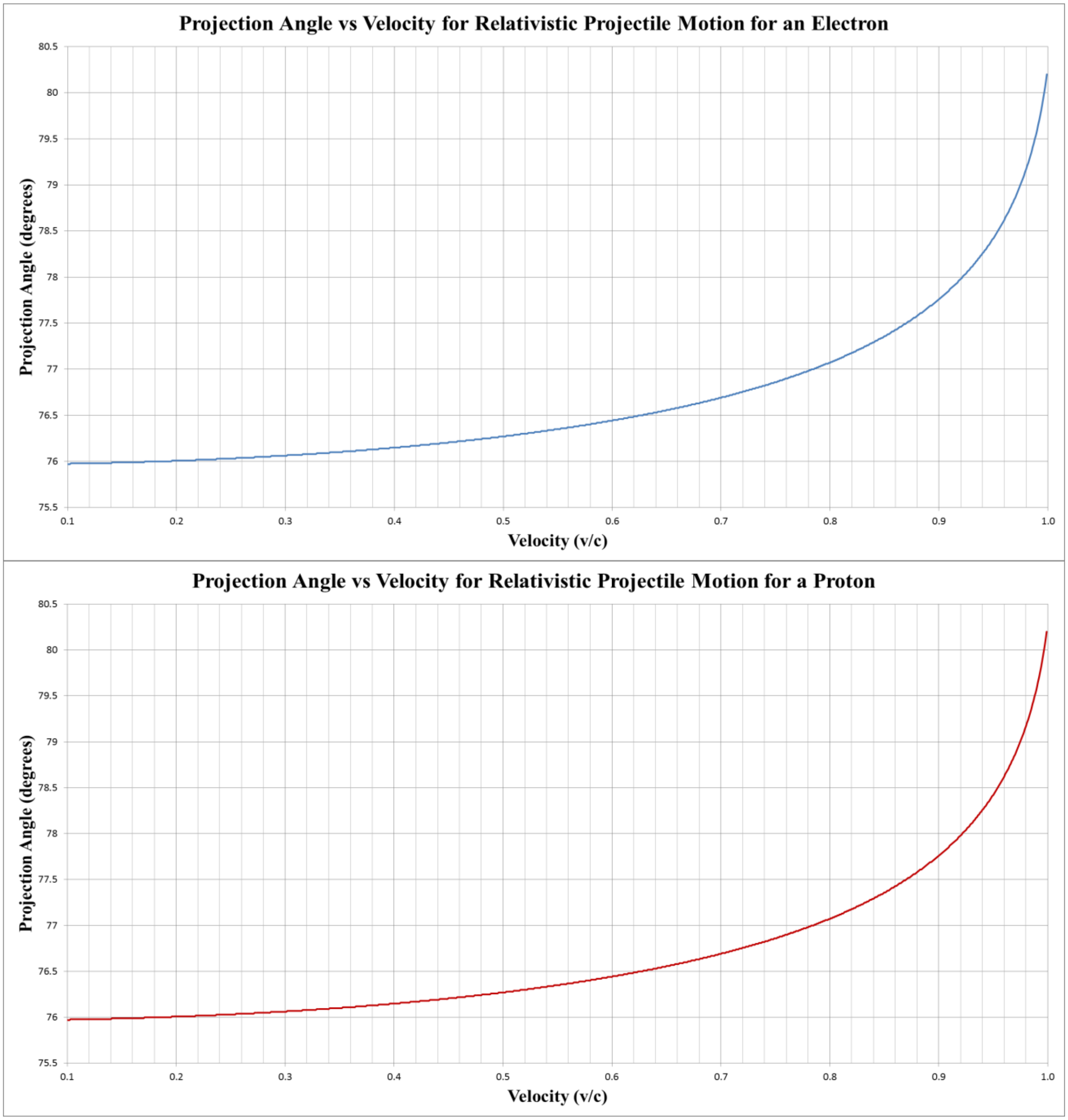


Figure 4. The plots of angles at which range and vertical height are equal as a function of velocity in terms of the speed of light for an electron (top) and proton (bottom). The curves are nearly identical, beginning around 76° at low speeds as expected, and rising sharply as v approaches c, reaching approximately 80.3° in both cases. This also shows that mass has no influence on projection angle, as expected.

Finally, to verify our results, the Range and Height equations (16) and (17) were reduced to non-relativistic equations where the speed approaches zero and where the speed approaches *c*. Using the algebraic approximation $ln(1+x)= x$ for small values of *x*, the Range and Maximum Height of the projectile were simplified to:

$$R = \frac{2c^2 p_o^2 sin\theta cos\theta}{E_o F} \quad (18)$$

$$H = \frac{p_o^2 \sin^2\theta c^2}{2E_o F} \quad (19)$$

When setting equations (18) and (19) equal to each other, the result is *tan(θ) = 4*, or *θ = 76º*. This is the angle for which Range equals Maximum Height for non-relativistic projectiles, as expected.

When the velocity of the projectile approaches the speed of light, rest mass can be neglected in equation (9) and thus:

$$E_0 = p_0 c \quad (20)$$

If equation (20) is substituted for E_o in the range equation (16) and maximum height equation (17) , the result is as follows:

$$ln\left[\frac{1+sin\theta}{1-sin\theta}\right] = sec\theta - 1 \quad (21)$$

Since it is not possible to solve this equation algebraically for *θ*, the above equation was split into two curves and plotted on the same graph. The intersecting point was found to be at 80.3º. This verifies our previous findings that the angle at which Range equals Maximum Height for an ultra-relativistic speed is approximately 80.3º. Figure 5 shows the plot of *R* and *H* as a function of θ as *v* approaches 0 and as *v* approaches *c*. As expected, as *v* approaches zero, the angle of projection approaches 76º, as is seen in nonrelativistic cases, and as *v* approaches *c*, the angle approaches 80.3º, which is seen in Figure 4 for both the electron and proton, further supporting our results.

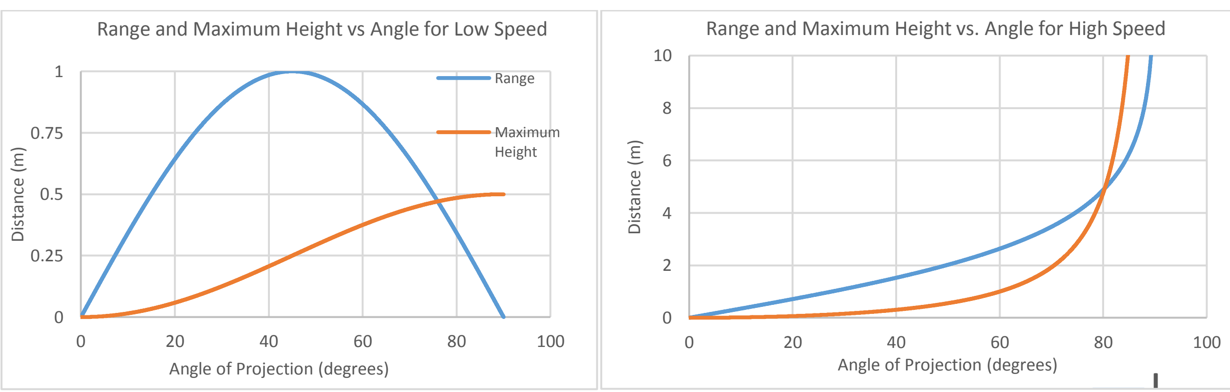


Figure 5. The plots of range and maximum height as a function of θ for when speed approaches zero (left). It should be noted that the left graph consists of the range and maximum height while the right graph does not. The second graph shows equation (17) broken into two curves on the same plot (right). The point at which the left side of equation (17) is equal to the right side of equation (17) is shown to be 80.3º.

It was found, both analytically and graphically that for relativistic projectiles, the angle of projection at which the Range is equal to the Maximum Height is not a constant angle of 76º, but rather a value that depends on the speed. We found that as the speed increases from non-relativistic to ultra-relativistic the angle changes from approximately 76º to approximately 80.3º. We also determined that the angle does not depend on mass, sign of charge, or the downward force, but is only dependent on the velocity of the projectile.

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