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Fractional Calculus and its Connection to the Tautochrone

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Abstract

Despite its brief mention in a letter written during the early days of classical calculus, Fractional Calculus remains a relatively untapped field. With most major contributions occurring in the last one-hundred years. In this paper, we will examine the fundamental aspects of Fractional Calculus and demonstrate how the modern definitions of the Fractional Integral naturally arise from solving the classic Tautochrone problem: finding a curve such that the time it takes an object to fall along this path is independent of its initial position. We then consider a generalization of the tautochrone by investigating the case when time is dependent on initial position. The result is a theorem building off the work published by Muñoz and Fernández-Anaya.

Keywords: Fractional Calculus, Mechanics, Mathematical Physics

1. Introduction and Fractional Calculus

The ideas of fractional calculus date back to the early days of calculus in the late 17th century, and while the theoretical foundations were laid down before the 20th century, the vast number of applications from groundwater flow to visco-elasticity have only been realized in the last 100 years. In this paper, our goals are to discuss the fundamentals of fractional calculus, study its connection to the tautochrone problem and , and then use this connection to study a generalization of the tautochrone.

Definition 1.1: Let $\gamma \in \mathbb{R}^+$, and Γ denote the standard Gamma Function, then the γth order *Riemann-Liouville integral*, denoted J_a^γ, is given by

$$J_a^\gamma f(x) := \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(t)}{(x-t)^{1-\gamma}}\, dt. \tag{1}$$

Example 1.1: We will calculate the γth order integral of the function $(x-a)^\beta$ with $\beta > -1$ and $\gamma > 0$. Using definition (1) it follows that

$$J_a^\gamma \left[(x-a)^\beta\right] = \frac{1}{\Gamma(\gamma)} \int_a^x (t-a)^\beta \,(x-t)^{\gamma-1}\, dt. \tag{2}$$

Let $s = (t-a)/(x-a)$, then $ds = dt/(x-a)$. Note that when $t = a$, $s = 0$, and when $t = x$ we find that $s = 1$. Combining like terms we arrive at

$$\frac{1}{\Gamma(\gamma)}(x-a)^{\beta+\gamma} \int_0^1 s^\beta (1-s)^{\gamma-1}\, ds. \tag{3}$$

The integral in (3) is in fact Euler's Beta Function[4], defined as

$$\int_0^1 s^\beta (1-s)^{\gamma-1}\, ds = \frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma+1)}. \tag{4}$$

Note that β and γ must be positive real numbers in order to apply Euler's Beta Function. Inserting equation (4) into equation (3) we conclude that

$$J_a^\gamma \left[(x-a)^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}(x-a)^{\beta+\gamma}. \tag{5}$$

An interesting characteristic of fractional calculus is the development of multiple definitions for the fractional derivative. In this paper we will focus on the Riemann-Liouville form.
Definition 1.2: Let $\gamma \in \mathbb{R}^+$, and $m = \lceil\gamma\rceil$. Where $\lceil\gamma\rceil$ is the ceiling funtion defined as the smallest integer greater than γ. Then the *Riemann-Liouville fractional derivative*, denoted D_a^γ, is given by

$$D_a^\gamma f := D^m J_a^{m-\gamma} f. \tag{6}$$

Using (6), we can calculate the γth order derivative of $(x-a)^\beta$. By Definition 1.2:

$$\begin{aligned}
D^\gamma (x-a)^\beta &= D^m J^{m-\gamma}(x-a)^\beta \\
&= D^m \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+m-\gamma+1)}(x-a)^{\beta+m-\gamma}\right] \\
&= \frac{\Gamma(\beta+1)\Gamma(\beta+m-\gamma+1)}{\Gamma(\beta+m-\gamma+1)\Gamma(\beta-\gamma+1)}(x-a)^{\beta-\gamma} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}(x-a)^{\beta-\gamma}.
\end{aligned} \tag{7}$$

Note that we applied the general power rule

$$D^m(x^k) = \frac{\Gamma(k+1)}{\Gamma(k-m+1)} x^{k-m} \tag{8}$$

where $k \in \mathbb{R}^+$, to compute D^m. One way to see if this formula makes any sense is to see what happens in the classical case. In particular, we can consider what happens when $\gamma = 1$. Then, according to (7),

$$\begin{aligned}
D^1(x-a)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta)}(x-a)^{\beta-1} \\
&= \beta(x-a)^{\beta-1}.
\end{aligned}$$

Which is exactly what we would expect.

2. The Tautochrone Problem and Fractional Calculus

Consider an object of mass m falling under the force of gravity constrained to a curve given by $\Psi(y) = x$ seen in Figure 1 on the next page. We would like to find what kind of curve makes the time for the particle to fall from y_* to some $y < y_*$ independent of y_*.

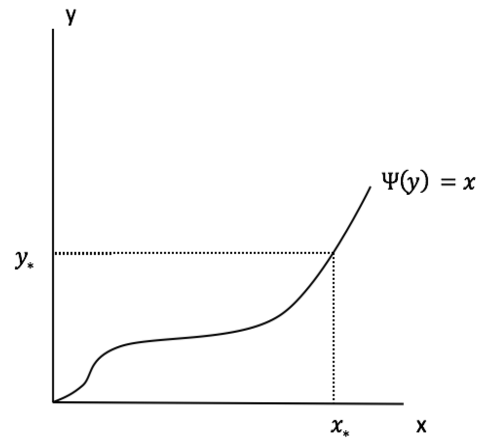

Figure 1: A possible path, $\Psi(y)$, for the particle to follow.

From the arc length formula, we know that

$$\frac{ds}{dy} = \sqrt{1 + \Psi'(y)^2}. \tag{9}$$

we also know that

$$-\frac{ds}{dt} = v. \tag{10}$$

with v representing the velocity of the particle. The time derivative of s is negative since the particle is falling as time increases. Since the curve we are examining is frictionless, we may apply conservation of energy. That is, $T(y_*) + U(y_*) = T(y)$. Where y_* is the particle's position at $t = 0$. It follows that,

$$v = \sqrt{2g(y_* - y)}.$$

Note that $y < y_*$. We may now rearrange (10) to solve for the time it takes the particle to travel from $y = y_*$ to $y = 0$:

$$dt = -\frac{ds}{v} \tag{11}$$

$$= -\frac{\sqrt{1 + \Psi'(y)^2}}{\sqrt{2g(y_* - y)}} dy. \tag{12}$$

For the moment we will let $\phi(y) = \sqrt{(1 + \Psi'(y)^2)/2g}$, and integrating (12) from $t = 0$ to

$t = t(y_*)$, and $y = y_*$ to $y = 0$ respectively we obtain:

$$\begin{aligned}\int_0^{t(y_*)} dt &= -\int_{y_*}^{0} \frac{\phi(y)}{\sqrt{y_* - y}} dy \\ &= \int_0^{y_*} \frac{\phi(y)}{\sqrt{y_* - y}} dy.\end{aligned}$$

Thus we arrive at an expression $t(y_*)$ for the time it takes the particle to travel from $y = y_*$ to $y = 0$. This expression is known as *Abel's Integral Equation of the first kind*:

$$t(y_*) = \int_0^{y_*} \frac{\phi(y)}{\sqrt{y_* - y}} dy. \tag{13}$$

Which is in fact the convolution of ϕ and $y^{-1/2}$, denoted as:

$$t(y_*) = \phi * y_*^{-1/2}. \tag{14}$$

The Laplace transform and convolutions have the very elegant property that the Laplace transform of a convolution is the same as the product of the Laplace transforms, i.e.

$$\mathscr{L}\{g * h\}(s) = \mathscr{L}\{g\}\mathscr{L}\{h\}. \tag{15}$$

We can apply this property to (14). Let $\mathscr{L}\{\phi\}(s) = \Phi(s)$, $\mathscr{L}\{t(y_*)\}(s) = T(s)$ and note that $\mathscr{L}\{C\,t(x)\}(s) = C\,T(s)$ for $C \in \mathbb{R}$. We also know that for $n \in \mathbb{R}$:

$$\mathscr{L}\{y^n\}(s) = \frac{\Gamma(n+1)}{s^{n+1}}. \tag{16}$$

Then it follows from (16) that the Laplace Transform of (14) is given by

$$T(s) = \Phi(s)\frac{\Gamma(1/2)}{s^{1/2}}. \tag{17}$$

Next we will isolate $\Phi(s)$ and take the inverse Laplace Transform of both sides:

$$\begin{aligned}\phi(y_*) &= \mathscr{L}^{-1}\left\{T(s)\frac{s^{1/2}}{\Gamma(1/2)}\right\}(y_*) \\ &= \mathscr{L}^{-1}\left\{sT(s)\frac{s^{-1/2}}{\Gamma(1/2)}\right\}(y_*). \end{aligned} \tag{18}$$

To obtain (18) we observe that $s^{1/2} = ss^{-1/2}$. Let $h = \mathscr{L}^{-1}\{s^{-1/2}\}(y_*)$. By (15):

$$sT(s)\frac{s^{-1/2}}{\Gamma(1/2)} = \frac{1}{\Gamma(1/2)}\mathscr{L}\left\{\frac{d}{dy_*}\left[t * h\right]\right\}(s). \tag{19}$$

To solve for h, observe the following:

$$
\begin{aligned}
\mathscr{L}^{-1}\{s^{-1/2}\}(y) &= \frac{1}{\Gamma(1/2)}\mathscr{L}^{-1}\Big\{\frac{\Gamma(1/2)}{s^{1/2}}\Big\}(y_*) \\
&= \frac{1}{\Gamma(1/2)}y_*^{-1/2}. \qquad (20)
\end{aligned}
$$

Upon substituting (20) into (19) we obtain

$$
\begin{aligned}
sT(s)\frac{s^{-1/2}}{\Gamma(1/2)} &= \frac{1}{\Gamma(1/2)\Gamma(1/2)}\mathscr{L}\Big\{\frac{d}{dy_*}\big[t * y_*^{-1/2}\big]\Big\} \\
&= \frac{1}{\pi}\mathscr{L}\Big\{\frac{d}{dy_*}\big[t * y_*^{-1/2}\big]\Big\}. \qquad (21)
\end{aligned}
$$

Finally we can substitute this result into (18):

$$
\begin{aligned}
\phi(y_*) &= \mathscr{L}^{-1}\Big\{\frac{1}{\pi}\mathscr{L}\Big\{\frac{d}{dy_*}\big[t * y_*^{-1/2}\big]\Big\}\Big\}(y_*) \\
&= \frac{1}{\pi}\frac{d}{dy_*}\big[t * y_*^{-1/2}\big] \\
&= \frac{1}{\pi}\frac{d}{dy_*}\int_0^{y_*}\frac{t(y)}{(y_* - y)^{1/2}}\, dy. \qquad (22)
\end{aligned}
$$

It is a well known fact that solving (22) when $t(y)$ is constant yields a cycloid curve. However, the more interesting note at this point is that the expressions in equations (13) and (22) look familiar. Compare the Riemann-Liouville Integral with Abel's Integral Equation

Riemann-Liouville Integral: $$\frac{1}{\Gamma(\gamma)}\int_a^x \frac{f(t)}{(x-t)^{1-\gamma}}\, dt$$

Abel's Integral Equation: $$\int_0^{y_*}\frac{\phi(y)}{\sqrt{y_* - y}}dy,$$

and (22) above with the Riemann-Liouville Derivative

$$D^m J_a^{m-\gamma} f.$$

By setting $\gamma = 1/2$, it can be seen that Abel's Integral equation is simply the half integral of the curve $\phi(y)$, which is the arclength of the path $\Psi(y)$. It follows that the inverse problem, solving for the curve that yields a given time function, is also the inverse operation. That is, to find the curve given a time curve, one computes the half derivative of the time function. The beautiful thing about this is that a concept as abstract as fractional calculus naturally arises in a classic problem in physics. Not only do the modern definitions arise, but the inverse relationship between the Riemann-Liouville operators also comes about naturally. This also allows mathematicians to talk about problems such as the tautochrone in a more precise and efficient manner. Instead of describing a complex integral equation, we can now simply say that we are taking the half integral of a curve or the half derivative of a time function. This will be useful in the following section.

3. Variable Time Functions

When solving the tautochrone problem, we consider time functions of the form

$$t(y) = k,$$

where $k \in \mathbb{R}$. A natural question to ask next is what if we let t vary with initial position y. Is there a way to test whether or not a given time function will yield a real valued and differentiable curve $x(y)$? The case when

$$t(y) = C(y - y_0)^{\beta}, \tag{23}$$

where C, $\beta \in \mathbb{R}$ and $y \in (y_0, y_*]$, for some y_*, has been studied[6] in depth. It was found that

$$-\frac{1}{2} < \beta \leq \frac{1}{2} \tag{24}$$

if our curve is to be real valued and continuous. We want to extend this idea to a wider range of functions, which motivates the following definition:
Definition 3.1: *An* ***Abel Function*** *is a function $t(y)$ such that*

$$x(y) = \int_{y_0}^{y_*} \sqrt{(ds/dy)^2 - 1}\, dy,$$

where

$$\frac{ds}{dy} = \frac{\sqrt{2g}}{\pi} D^{1/2}[t(y)],$$

is a real valued, continuously differentiable, bounded curve in the yx-plane on which a particle can fall under the force of a uniform gravitational field.

We begin this investigation into Abel Functions by considering time as a power series in the following proposition,
Proposition 3.1: *If $t(y)$ is such that,*

$$t(y) = \sum_{j=0}^{\infty} \alpha_j (y - y_0)^{\beta_j}, \tag{25}$$

where $\alpha_j, \beta_j \in \mathbb{R}$, $\beta_{min} \in (-1/2, 1/2]$ and

$$\mu(y) = \sqrt{\frac{2g}{\pi}} \sum_{j=0}^{\infty} \alpha_j \frac{\Gamma(\beta_j + 1)}{\Gamma(\beta_j + 1/2)} (y - y_0)^{\beta_j - \beta_{\min}}$$

is a bounded series and $\mu(y) \geq (y - y_0)^{1/2 - \beta_{\min}}$ for all y in the interval $I = (y_0, y_*]$. Then $t(y)$ is an Abel Function on I.

Proof. Suppose

$$t(y) = \sum_{j=0}^{\infty} \alpha_j (y - y_0)^{\beta_j},$$

satisfies all conditions of Propsition . Substituting $t(y)$ into (22) we find that

$$\begin{aligned}\phi(y) &= \frac{ds}{dy} \\ &= \frac{\sqrt{2g}}{\pi} \frac{d}{dy} \int_{y_0}^{y} \frac{\sum_{j=0}^{\infty} \alpha_j (y' - y_0)^{\beta_j}}{(y - y')^{1/2}} dy'.\end{aligned}$$

Let $\nu = y' - y_0$, and therefore $d\nu = dy'$,

$$\frac{ds}{dy} = \frac{\sqrt{2g}}{\pi} \frac{d}{dy} \sum_{j=0}^{\infty} \alpha_j \int_0^{y-y_0} \frac{\nu^{\beta_j}}{(y - y_0 - \nu)^{1/2}}\, d\nu.$$

Once again we come across a convolution, and we will use the same methods from Section 2 to evaluate it. We then have

$$\begin{aligned}\frac{ds}{dy} &= \sqrt{\frac{2g}{\pi}} \sum_{j=0}^{\infty} \alpha_j \frac{\Gamma(\beta_j + 1)}{\Gamma(\beta_j + 3/2)} \frac{d}{dy} (y - y_0)^{\beta_j + 1/2} \\ &= \sqrt{\frac{2g}{\pi}} \sum_{j=0}^{\infty} \alpha_j \frac{\Gamma(\beta_j + 1)}{\Gamma(\beta_j + 1/2)} (y - y_0)^{\beta_j - 1/2}.\end{aligned}$$

To ensure that $s(y)$ is finite, we require $\beta_j > -1/2$ for each β_j. Without loss of generality, suppose $\beta_0 = \min(\beta_j)$, and that $\beta_0 < 1/2$. It follows that,

$$\begin{aligned}\frac{ds}{dy} &= \frac{(y - y_0)^{1/2-\beta_0}}{(y - y_0)^{1/2-\beta_0}} \sqrt{\frac{2g}{\pi}} \sum_{j=0}^{\infty} \alpha_j \frac{\Gamma(\beta_j + 1)}{\Gamma(\beta_j + 1/2)} (y - y_0)^{\beta_j - 1/2} \\ &= \frac{1}{(y - y_0)^{1/2-\beta_0}} \sqrt{\frac{2g}{\pi}} \sum_{j=0}^{\infty} \alpha_j \frac{\Gamma(\beta_j + 1)}{\Gamma(\beta_j + 1/2)} (y - y_0)^{\beta_j - \beta_0}.\end{aligned}$$

Therefore,

$$\left(\frac{ds}{dy}\right)^2 = \left[\sqrt{\frac{2g}{\pi}} \frac{\sum_{j=0}^{\infty} \alpha_j \frac{\Gamma(\beta_j+1)}{\Gamma(\beta_j+1/2)} (y - y_0)^{\beta_j - \beta_0}}{(y - y_0)^{1/2-\beta_0}}\right]^2.$$

Let

$$\mu(y) = \sqrt{\frac{2g}{\pi}} \sum_{j=0}^{\infty} \alpha_j \frac{\Gamma(\beta_j + 1)}{\Gamma(\beta_j + 1/2)} (y - y_0)^{\beta_j - \beta_0}.$$

We want to ensure that

$$\frac{dx}{dy} = \sqrt{\left(ds/dy\right)^2 - 1}$$

remains real valued for all $y \in I$. By the conditions of Proposition 1.1,

$$(y - y_0)^{1/2-\beta_0} \le |\mu(y)| < M.$$

for all $y \in I$, and $M \in \mathbb{R}^+$. It follows that,

$$\left|\frac{\mu(y)}{(y - y_0)^{1/2-\beta_0}}\right| \ge 1,$$

on I. Therefore $t(y)$ is an Abel Function on I. □

Our goal is to use Proposition 3.1 to analyze functions which can be represented as Taylor Series, and test whether or not they are Abel Functions. We will also make use of the following proposition in proving Theorem 1:

Proposition 3.2: *If*

$$\sum_{j=0}^{\infty} \frac{\Psi^{(j)}(y_0)}{\Gamma(j+1)}(y - y_0)^j \tag{26}$$

converges absolutely with radius of convergence $\rho > 0$, then

$$\sum_{j=0}^{\infty} \frac{\Psi^{(j)}(y_0)}{\Gamma(j+1/2)}(y - y_0)^j \tag{27}$$

converges absolutely with radius of convergence $\rho' = \rho$.

With Propositions 3.1 and 3.2, we can prove the following theorem:

Theorem 1. *A function $\Psi(y)$ is an Abel Function on some interval I if*

$$\Psi(y) = \sum_{j=0}^{\infty} \frac{\Psi^{(j)}(y_0)}{\Gamma(j+1)}(y - y_0)^j, \tag{28}$$

and $\Psi(y_0) \neq 0$.

Proof. Let $\Psi(y)$ be a function given by equation (28) with $\Psi(y_0) \neq 0$. If we can show that $\Psi(y)$ satisfies Proposition , then we are done. Therefore we will begin by using the methods from Section 2 to compute the following,

$$\begin{aligned}
\frac{ds}{dy} &= \frac{\sqrt{2g}}{\pi} D^{1/2}\,\Psi(y) \\
&= \frac{\sqrt{2g}}{\pi} DJ^{1/2}\,\Psi(y) \\
&= \frac{\sqrt{2g}}{\pi}\frac{d}{dy}\int_{y_0}^{y} \frac{\sum_{j=0}^{\infty}\frac{\Psi^{(j)}(y_0)}{\Gamma(j+1)}(y'-y_0)^j}{(y-y')^{1/2}}\,dy' \\
&= \sqrt{\frac{2g}{\pi}}\sum_{j=0}^{\infty}\frac{\Psi^{(j)}(y_0)}{\Gamma(j+1/2)}(y-y_0)^{j-1/2},
\end{aligned}$$

Note that $\beta_{\min} = 0$. That is, the smallest power in the Taylor series representation of time is 0 since the lowest index of the series is $j = 0$. Bringing the constants out of the sum,

$$\frac{ds}{dy} = \frac{1}{(y-y_0)^{1/2}}\sqrt{\frac{2g}{\pi}}\sum_{j=0}^{\infty}\frac{\Psi^{(j)}(y_0)}{\Gamma(j+1/2)}(y-y_0)^j.$$

In this case,

$$\mu(y) = \sqrt{\frac{2g}{\pi}}\sum_{j=0}^{\infty}\frac{\Psi^{(j)}(y_0)}{\Gamma(j+1/2)}(y-y_0)^j.$$

By Proposition 3.2, $\mu(y)$ converges absolutely. Therefore it is bounded above. Next, note that

$$\begin{aligned}
|\mu(y_0)| &= \frac{\sqrt{2g}}{\pi}|\Psi(y_0)| \\
&> 0 \\
&= (y-y_0)^{1/2}\big|_{y=y_0}.
\end{aligned}$$

Hence there exists at least one interval $I = (y_0, y_*]$ such that

$$(y-y_0)^{1/2} \le |\mu(y)| < M.$$

Thus, by Proposition 3.1, $\Psi(y)$ is an Abel Function on I. □

It is interesting to note that our investigation into Abel Functions involves the experimentally measured value of g at sea level: $9.81\,m/s^2$. Since $\sqrt{2g}/\pi > 0$ for any value of g given by

$$g = \frac{G\,m}{R^2},$$

where G is the universal gravitational constant and R is the distance from the mass's center of mass, Abel Functions exist in any gravitational field with magnitude given by g.

4. Conclusion

Fractional Calculus remains a valuable tool in gaining a new perspective on classic problems. When applied to a problem as old as the Tautochrone, we gain a more precise, and arguably more aesthetically pleasing, way to discuss the problem. We go from integral equations involving time and arclengths to half integrals and half derivatives of arclength and time respectively. There is a certain symmetry in this perspective that mirrors the problem as well. We have the inverse problem of finding a curve given time, and we solve this problem by applying the inverse differential operator. Moreover, questions that may never have been asked arise very naturally in this new perspective. For instance we could now consider Abel Functions where we apply the 1/4th derivative, the πth, or more exotic orders. These questions can lead to further interesting investigations, and they become a natural next step when we reframe the problem in terms of Fractional Calculus.

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