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"Is Math Real? A Critical Examination of Mathematical Realism and Empiricism"

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Abstract

Are numbers, like language, a mere psychological construction, useful and only true inasmuch as human manners of thinking apply to it? Or is mathematics a truly universal language, a form of reality independent of human thought? To answer those questions, one must bring philosophy and mathematics to the table. This paper utilizes primarily an historical approach to first, examine the central features of Platonic realism, which claims math is real (math holds an objective reality-status and is independent of human thinking), and second, examine the central features of nominalistic empiricism which claims math is non-real (math is merely a psychological construction based upon the perception of sense-data furnished by a physical-material nature; math's reality status is hence subjective and depends upon the activity of the perceiving mind). Then the author will critically examine the two approaches by reviewing two major philosophers' views. Using the proof concerning the infinite amount of prime numbers as a test case, one may attempt to use the concept of infinity as such to examine both empiricism and Platonism. In the conclusion of the study, it is shown that only objective and independent reality of math's existence can solve the dilemma that arises in the empiricist approach.

Keywords: Realism, Numbers, Mathematical Philosophy

1. Introduction

Galileo once said, "The Book of nature is written in the language of mathematics."[1] Is math, like language, a mere psychological construction, useful and only true inasmuch as human manners of thinking apply to it? Or, is mathematics a truly universal language, a form of reality independent of human thought? Philosophers have long debated the nature of mathematics throughout history. This was not without real benefit or discovery. Rationalists, from Plato, to Leibniz, were all impressed by the nature of mathematics and questioned what constitutes "pure" rationality and mathematical reason. For them, mathematical knowledge seems to be based on reason's power to engage in proof, provide axioms, and deductively solve for necessary conclusions. The truths of mathematics were said to be "ideal," mind-like or immaterial, but certainly independent of and true apart from the human mind. They proceed purely through reason, not empirical observation, which is a fundamental part of natural science. On the other hand, there are empiricist philosophers, like John Stuart Mill, who believed that math is just a pure generalization and construction of the human mind, based only upon empirical sense data provided by experience. Nature, the empiricists thought, does not admit of ideal or immaterial elements, as had thought the rationalists. Nature is purely physical or material and ideas could only be explained in terms of their dependence upon physical or material properties, sense-data. In this paper, the author used an historical approach to examine the central features of Platonic realism and Mill's empiricism. Plato claims mathematics references something real beyond empirical observations, beyond mere constructions created by the mind based upon empirical generalizations. Mill claims mathematics references only psychological processes of the mind which can be explained in terms of sense experience. Which perspective is best able to explain the real nature of mathematics?

2. Hypothesis

Platonic realism offers a better explanation than the central features of mathematical empiricist, which claims that the reality of mathematics is non-real: that the objects of mathematical ontology exist only in the mind as nominalistic constructions created from generalizations created by the mind. These generalizations are necessarily always tied to the physical-material world for without sense-experience of the world there can be no data upon which the mind can work and from which it might generalize. The approach of Plato's realism offers a better account of mathematical truth than does mathematical empiricism, the author here argues.

3. Philosophical Background

3.1. Why Does It Matter?

Why does the question of mathematical realism matter? A non-mathematician might ask, "Why should anyone care about all of this?" When one studies math, they will give precise definitions for any abstract concept that they are going to introduce. For example: one can define complex number using two real numbers with *i* as a solution of the equation $x^2 = -1$. Someone can also use the quotient or fraction of two integers to define and express a rational number. But then it gets ambiguous to provide a definition for $\mathbb{N} = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ... \}$ or numbers that occur commonly in nature. Using such a mathematical approach, one enters into the philosophical realm of questions such as: What is a number? What is an integer? [2] What does it mean to say that there is no end to the number pi? Or the most fundamental question; what is mathematics the study *of* exactly?

Although they are simple questions, it is hard to give an answer if one is not *philosophically* aware of the nature of mathematics - whether what mathematics *does* references something *real* or whether mathematics is simply a useful, but non-reality matching *tool* of the human brain. If the former mathematics attains *truth*. If the latter the goal of mathematics is merely *function*, but without regard for truth (and thus truth becomes relative to the procedure).

Here are four reasons that the debate over the objectivity of mathematics is important. First, it determines the location of math in academia, especially determining whether math is part or not part of natural science. Second, whether math is an *accurate* tool for building descriptions of things or is it fundamentally a description itself. Third, if math is real, then it has to be necessarily true, and does not merely happen to be true just by circumstance. Fourth, as Gödel said, "The question of the objective existence of the objects of mathematical intuition … is an exact replica of the question of the objective existence of the outer world."[3] If math is real, then scholars can find answers to these questions.

3.2. Plato's Realism

Realism, in ontology, is the view that at least some mathematical objects exist objectively and that objectivity independent of the human mind and its actions accounts for the necessity of mathematics and its subject matter.[4] There are distinctive subject matters that are independent of the mathematician, such as mathematical objects, according to mathematical realism: functions and relations, species and genera or "kinds" of number, "kinds" of infinity, and so on… Sometimes called Platonism, mathematical realism admits causal, eternal, indestructible, necessary mathematical objects that are not part of space and time and not subject to the physical laws of the physical-material universe.[4] In this sense, Plato would claim they are "ideal" - that is, immaterial (although for him, as the mind, too, is immaterial, it is about to think about them). Natural science is unable to enter into or test or describe the laws and objects of mathematical realism since its methods cannot describe or detect the presence of number or determine numbers' properties given the realists characterizations of mathematics' general reality-status, its ontological nature.

Plato is the first and foremost mathematician who attempted to prove that what the practice of mathematics references is objectively true and real using his theory of forms. In his theory, there is a realm of forms, or ideal patterns of meaning, Ideals, present in the other eternal world called world of being. This world is immaterial, eternal, acausal, absolutely true, necessary, and independent from the physical world, and accessible only through the activity of thought - the "motion" of the soul known as reason. Forms are organized and ranked into the ultimate Ideal, what

Plato calls “The Good,” below that the Forms of virtues and the transcendentals (truth, beauty, goodness, and justice), forms of physical objects (as essential natures, so “chariness” for a physical chair; “appleness” for an apple, and so on), and forms or patterns of mathematical and geometrical objects. As a soul is carried from the physical world up toward the world of being, we have the mental image which recognizes the perfect, ideal world and its contents which are concepts represented in the physical world.[4] That is, the activity of the soul - reason - occurs by means of thinking about concepts, or ideas which reference the forms. All physical objects have defects, and we can see when an object resembles the its form in physical life. Similarly, he holds that proposition from arithmetic, geometry and algebra are true or false, that they exist in the world of being, independent of the mathematician, the physical world, and the mind, thus, mathematics is just recollection of what we have seen from the world of being.[5]

In the physical world, one cannot find any line, circle, point that strictly meets such a definition, i.e. line without width, points without dimensions. It is true that the axioms based theorem is the practice of using constructive dynamic language in math has not been changed in math since Euclid. However, Plato believed this way of “language” is nonsense.[4] For example, Plato would say that there is a straight line between any two points while Euclid would say between any two points one can draw a straight line. So, which view is correct? This is a chicken or egg question that leads to no solution. Gödel made a defense of the insolubility of the question by saying circular logic is human dependent because of mental limitations. It clearly exists as a “definition” outside of human construction, and concepts may be conceived as real objects existing independently of humanity and human definitions and constructions. Gödel developed the first incompleteness theorem, stating that, "Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F."[8] In his proof he used some properties of set theory to show that the mathematical realm is independent of the sensory experience, thus math is a priori par excellence.[4]

To sum so far, Plato believed that the propositions of mathematics are objectively true or false, thus independent from mind and language. Mathematical objects are real. And mathematical knowledge exists independently of us. Although his theory of forms may be outdated and may be rejected by later philosophers, his realist ontology which holds that mathematical knowledge is a priori, is a valid answer given the characterization of mathematical knowledge. As Frege, a Platonist himself, stated, “Pythagorean theorem (mathematical theorem) is timelessly true, true independently of whether anyone takes it to be true. It needs no bearer. It is not true for the first time when it is discovered, but is like a planet which, already anyone has seen it, has been in interaction with other planets.”[2] This belief strongly contrasts with the empiricists.

3.3. Mill’s Empiricism

John Stewart Mill, who was born 2200 years later than Plato, represents the opposite of the Platonists stance. He holds an absolute empirical view regarding the nature of mathematics known as “nominalism”. He is different from Kant, a rationalist who sought to synthesize empiricism with rationalism by considering the propositions of logic as real and thus synthetic and empirical. Mill rejected Kant’s attempt to synthesize rationalism and empiricism and argued that math is strictly empirical and philosophy reveals the presupposition of empirical science. Science bases its findings upon a physical-material world. Humans perceives that world and the world furnishes the human mind with sense-data. Sense-data is processed by the mind and forms the psychological contents of subjective human experience. As such, Mill's empiricism is thus that of a consistent naturalist, stating that the human mind is part of nature, thus no knowledge of the world can be a priori.[4] No sense-data? Then no experience in the mind. Mill holds that human perception and intuition concerning mathematics are an empirical matter only and absolutely nothing else. Mathematics, for Mill, is thus a matter of subjective human agreement concerning how the mind generalizes laws about the sense-data it perceives. His work influences till this day many important contemporary empiricist accounts of mathematics.

Mill believed that all laws of mathematics are derived from enumerative induction - an inference from observation via generalization on what is observed.[4] For example, if one can only see black cows and no white cows, we will conclude that it is true that all cows are black and the next cow we see will be black. The reason is based on what we have observed and what we expect to observe. Similarly, Mill applies experimental enquiry in science in order to construct a circuitous argument using enumerative induction between the scientific laws and the generalization process from experience. For him, mathematic knowledge is based on observation of existing subjects and indirectly traced back to human generalization. He further argued that inference is from ‘particulars to particulars’ and all mathematical propositions are generalizations which record and summarize experience.

Mill rejected the existence of abstract objects, and sought to found a geometry of observation. He stated that the geometric objects are approximations, thus geometry concerns the idealization of the possibility of construction. To prove his point, he used the idea of limit concepts: lines being without breadth and points without length, all limit the approach on how people can draw upon given the limit they have with their tools. He concluded that geometry is a work of fiction since it does not deal with existing objects, they are only approximately drawn figures to the extent of that real figure they purport to represent. Such approximation is just idealization. He rejects axioms, e.g. the angle of triangle is formed about a two-right angle depend on how the lines are drawn. Thus, geometry is inductive generalization about possible physical figures in space.

3.4. Challenges Faced Using The Empiricist Approach

Mill's empiricist (nominalist) view of the shortfalls of mathematical realism faces several challenges. First, is a generalization even true? Are there subatomic lengths or limitations for the postulate? His ideas concerning physical possibility rather than mathematical possibility lead to an issue where consistency breaks down under physical laws of the universe. However, there is no limit in the mathematical world which creates mathematical possibilities.

Based on his empiricist standpoint, natural numbers are just a number of collections, thus he rejects the existence of ideal 'units' - the ontological reality of "number." This conclusion is derived from his statement that math is a generalization based on observation, where "all numbers must be numbers of something; there is no such things as numbers in the abstract."[4] For Mill, all numbers are just a number of collections of a number of ordinary collections. For example, "one" means one apple, one candy, one of some empirically observed physical thing or "unit." He holds to axioms where "things which are equal to the same thing are equal to one another" and "equals added to equals make equal sums", based on these two axioms he holds that 5+2=7.[4]

However, when you take arithmetic and apply Mill's view to large numbers, the empiricist's reasoning falls apart. For example, how and where does generalized experience come from for 21312124 + 12312313 = 33624437? When will a person ever encounter an experience of such large number where there are 21312124 balls in one pile and 12312313 balls in another pile? Clearly there is no need to add them one by one and determine the number. If people attempts to go through the process, then very few of them are able to get the correct answer.[4] If it is bees instead of balls, it would take more time and they would get more inaccurate results (one may get stung by some bees and lose count). Frege uses the example of adding two gallons of two liquids, where a chemical reaction or evaporation might change the volume of either or both of the two liquids, and thus it is impractical to use observed experience to confirm arithmetic sums.

Plato's argument concerning the fiction of geometry, is straightforward. As was written above the gate of the Academy, "Let no one ignorant of geometry enter here."[6] For Plato, if one is to come closer to truth, one actually ascends further from physically observed sense experience toward immaterial, ideal eternal truth. In fact, physical experience of the phenomenal world can never touch the ideal, immaterial truths of mathematical realities which exist in the nominal world. For example, using Plato's ontology, take the theorem that the tangent line to a circle intersects the circle at a single point. No matter how carefully one draws using whatever tool they might have, one will still see that the line overlaps with the boundary of the circle in a small portion. Empiricists may argue that the theorems of math do not exist since they fail to show that the intersection of a circle and a tangent line is a single point. Plato's explanation, however, is that the drawn lines and circles are just poor approximations on the real circle and real line, which is grasped only with the mind, the mind recognizes the real circle and the real line in the world of forms will intersect at a single point, which eternally exists in the world of being.

4. Proof

One simple and elegant mathematical proof to discount Mill's view is the Euclidean theorem and proof which shows that there are infinite prime numbers. The proof is done without import from any empirical experience and leads to a contrapositive statement that determines the truth of the statement. Suppose, contrary to the theorem, that there is only a finite number of primes. Thus, there will be a largest number called p. Now, define a number n as 1 plus the product of all the primes: Is n itself prime or composite? If it is prime then the original supposition is false, since n is larger than the supposed largest prime p. Now consider the composite of it. This means that it must be divisible (without remainder) by prime numbers. However, none of the primes up top will divide n (since there would always be a remainder of 1), so any number which does divide n must be greater than p. This means that there is a prime number greater than p after all. Thus, whether n is prime or composite, the supposition that there is a largest prime number is false. Therefore, the set of prime numbers is infinite, as show in statement (1).

$$n = (2 \times 3 \times 5 \times 7 \times 11 \ldots x\,p) + 1 \quad (1)$$

Theorem: There are infinitely many prime numbers.[2]

Mill's empiricist view also does not take account for many mathematical concepts that cannot be supported by experience. For instance, the concept of infinity, where Cantor's theory provided a controversial proof that not only defined the nature of infinity, but it also proved that multiple infinities existed, and some were larger than others. With geometry, where Mill concluded as a work of fiction using idealization and approximation, he forgot that most mathematical propositions do not conform to experience. In fact, when math is studied, empirical experiences are often ignored instead the focus is on ideal concepts. But as Plato claimed, ideal concepts are not just mental, they are extra-mental and objectively real and true (or otherwise this proof would not force thinking to conform to it; the proofs would conform to the mathematician's thinking which is not the case).

5. Conclusion

Philosophers were among the first mathematicians: they were among the first intellectuals that attempted to understand the world. The very idea of "mathematics" underwent profound transformation and development as various philosophers - whether Pythagoras, Plato, Aristotle, Descartes, Leibniz, Whitehead, or today Alain Badiou - work to define what exactly mathematics is and what it does. Math is a case study for philosophers who take a rationalist approach and wish to demonstrate, contra the empiricists, that the reality mathematics references is not one strictly dependent upon the human mind and not existent merely for communal purposes as a contingent tool among others. The *reality* referenced by mathematics allows philosophy to make formulations that focus on logic (reasoning), model theoretic semantics, ontology, set theory, and countless more. Through the development of empirical science since the 17th century in particular, the realm of mathematic has expanded from algebra and geometry to almost every subject. Today, mathematics touches nearly every aspect of human life. Therefore, it is important to understand not just what mathematics *does*, but what mathematics *is.* Doing so will enable scholars to further expand the practical scope of mathematics, locking in what it is able to apply to and what it cannot. While the land of philosophy is getting smaller, as it shifts focus from the world or the universe itself to how humans know about the world - that is, a focus from the tradition of Continental philosophy and its concern with the normative dimensions of all of human life and the universe, to analytical philosophy, which focuses on language, logic, and mind (at the expense of thinking about the reality of logic, language, or mind - analytic philosophy is largely nominalist in character) - mathematical realism is more or less assumed and not defended by today's contemporary philosophers. In addition, it offers a better answer than the empiricists' view concerning the nature of mathematics and its application to daily life. This research has found that mathematical realism offers a better account of the reality of mathematical objects and truth than mathematical empiricism. Hopefully, this research might convince some to reconsider Platonic realism as they think about the reality of mathematics.

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