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Colliding-Wind Binaries with Orbital Motion: Line Wind Formulation

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Abstract

We apply the line wind approximation to thin shell bow shock and wind collision problems. Stellar wind collisions are observed in many binary star systems, where the orbital motion leads to the development of a spiral pattern. We discuss initial results for the shape of the shock structure in the co-rotating frame, which requires the inclusion of Coriolis and centrifugal effects. We only consider the problem for systems where the two winds are identical and the wind speeds greatly exceeds the orbital speed. Numerical integration of differential equations gives the shell shape.

Keywords: shock waves, stars: mass loss, hydrodynamics

1. Introduction

Binary star systems are relevant in astrophysics and much work has been done to observe them.[1] In binary systems, two stellar winds interact to develop a shocked structure. A common example of a colliding-wind binary is the system WR-104. The system consists of a Wolf-Rayet star and a OB main sequence star. The shocked structure has an observed nearly Archimedian spiral shape.[2] The theory of colliding-wind binaries is ongoing, and in many cases requires the use of 3-D numerical hydrodynamic simulations.[3–5] Even so, the theory is in development and simple analytic models for CWB shocks have yet to be found when orbital motion is included.

Analytic models for bow shock and wind collisions have been found for simplified geometries under the approximation of a thin shell.[6–8] The assumption of a thin-shell requires that we have efficient cooling of the winds when they collide. If the post-shock density is sufficiently high, the post-shock cooling can be efficient which leads to a small separation between the two shocks.

This paper examines the problem of an orbiting binary star system with supersonic wind collision, and presents initial results for the shell shape. We apply the simplified geometry of cylindrical winds in Section 2 to solve for the shape of the line wind bow shock. In Section 3, we describe the binary wind collision problem with orbital motion for winds with equal momentum loss rates and equal wind speeds.

1.1. Introducing the Line Wind

We use the concept of a line wind to simplify the geometry of the two wind collision problems discussed in this paper. A line wind emits fluid radially away from a vertical line of infinite extent, exhibiting both cylindrical and plane symmetry. Cylindrical winds have been considered in other wind collision problems, like 2D simulations.[4,9] We define the mass-loss rate per unit length per azimuthal angle of a line wind as λ. If we visualize a coaxial cylindrical Gaussian surface of length L and area A enclosing the line, we can interpret this parameter as $\lambda = \mathcal{F}\ A/2\pi L$, where $\mathcal{F}$ is the mass flux through the Gaussian surface. We write the density of a line wind as,

$$\rho = \frac{\lambda}{rv}, \tag{1}$$

where v is the launch speed of the wind and r is the cylindrical radius measured from the line wind. Because the line wind exhibits plane symmetry, we choose to just describe the winds for a thin slice in the orbital plane.

2. Line Wind Bow Shocks

In this section, we consider the collision of a cylindrical wind with a uniform ambient medium, accounting for the supersonic motion of the line wind through space with respect to the medium. Wilkin (1996, now W96) constructed analytic solutions for the shape of stellar wind bow shocks.[10] Cantó, Raga, & Wilkin (1996, hereafter CRW96) used algebraic equations to solve for the same bow shock shape. We use the latter approach to solve for the W96 bow shock solution algebraically, using the line wind to simplify the geometry of the problem.[11] The line wind is located at the origin in the yz-plane and moves through space with a constant speed v_* in the z-direction. We consider the problem in the frame of the line wind because it should be steady state in this frame. In this reference frame, the ambient medium is a wind with velocity $\vec{v}_* = -v_* \hat{z}$. The emitted wind collides with the ambient medium on the z-axis at a radius, referred to as the stagnation point, determined by balancing ram pressures, $\rho_w v_w^2 = \rho_a v_*^2$, which yields,

$$R_o = \frac{\lambda\, v_w}{\rho_a v_*^2}, \tag{2}$$

where the subscript w represents the line wind and the subscript a represents the ambient medium. The stagnation point radius, R_o, serves as the unit of length, and scales the shape of the bow shock in non-dimensional form.

2.1. Flux Function Description of Momentum Conservation

The condition of steady state requires that the flow rates of mass and momentum within the shell are equal to the mass and momentum incident upon the shell from both winds integrated from the stagnation point to a polar angle θ.[11] The angle θ is measured counterclockwise from the axis of symmetry (z-axis) to a point on the shell. Therefore, the quantities Φ_y and Φ_z are the momentum rates in the y and z directions through a coaxial cylinder with height Δx at an angle θ from the stagnation point, and Φ_j is the angular momentum flux through the same coaxial cylinder. The mass flux received on the shell through the coaxial cylinder integrated from the stagnation point to the angle θ is Φ_m. It should be noted that Φ_m differs from the mass flux, $\mathcal{F}$, used to define the mass loss rate of a line wind in Section 1.1. The quantity Φ_m is the mass flowing within the shell, which should not be confused with the mass flow from the line wind, $\mathcal{F}$, because Φ_m includes additional mass from the ambient medium. CRW96 use momentum flux conservation arguments to determine the spherical radius of a wind collision surface,

$$R = \frac{\Phi_j}{\Phi_y \cos\theta - \Phi_z \sin\theta}, \tag{3}$$

where the variables on the right side of the equation are functions of θ and R, and therefore describe the spherical radius at any position along the shell.[11]

2.2. Formulation and Solution of the Flux Functions

Here we derive the mass and momentum imparted on the shell by the wind and ambient medium. Because the coasting wind is momentum conserving, the surface integral is independent of the shape of the shell. Therefore, we choose to perform our integral over a coaxial cylinder with height Δx and a normal vector that is radial ($\hat{n} = \hat{r}$). The mass sent into the shell by the stellar wind is the same as the mass sent onto the coaxial cylinder from the direction of the symmetry axis through the angle θ,

$$\begin{aligned} \Phi_{w,m}\,\Delta x &= \int_0^\theta \rho_w \vec{v}_w \cdot \hat{n}\, dA, \\ \Phi_{w,m} &= \int_0^\theta \lambda\, d\theta = \lambda\theta, \end{aligned} \tag{4}$$

where the subscript m denotes the mass flux. We have written the integral in a form comparable to W96 Equation 4. We find the momentum propelled into the shell by the wind using the same formalism shown in our Equation 4. We consider $\hat{r}$ in cylindrical polar components. The integral for $\Phi_{w,z}$ is

$$\Phi_{w,z}\,\Delta x = \int_0^\theta \rho_w (\vec{v}_w \cdot \hat{n})(\vec{v}_w \cdot \hat{z})\, dA,$$

$$\Phi_{w,z} = \int_0^\theta v_w\, \lambda \cos\theta\, d\theta = \lambda v_w \sin\theta. \tag{5}$$

Similarly, the y-component and angular momentum fluxes are written

$$\Phi_{w,y} = \int_0^\theta v_w\, \lambda \sin\theta\, d\theta = \lambda v_w (1-\cos\theta), \tag{6a}$$

$$\Phi_{w,j} = 0. \tag{6b}$$

The angular momentum flux from the line wind $\Phi_{w,j}$ is zero due its location at the origin. Similarly, for the ambient medium we find,

$$\Phi_{a,m} = \int_0^y d\Phi_{a,m} = \int_0^y \rho_a v_*\, dy = \rho_a v_* y, \tag{7a}$$

$$\Phi_{a,z} = \int_0^y -\rho_a\, v_*^2\, dy = -\rho_a v_*^2 y, \tag{7b}$$

$$\Phi_{a,y} = 0, \tag{7c}$$

$$\Phi_{a,j} = \int_0^y \rho_a\, v_*^2\, y\, dy = \frac{1}{2}\rho_a v_*^2 y^2. \tag{7d}$$

The y- component of momentum flux from the ambient medium is zero because the ambient medium has no y-component of velocity. Inserting Equations 5, 6, and 7 into Equation 3 to solve for R we obtain,

$$R = \frac{2\,R_o}{1+\cos\theta}. \tag{8}$$

Equation 8 is an analytic solution for the spherical radius of the bow shock as a function of θ. A comparison with W96 Equation 9 (or CRW96 Equation 18) reveals that the line wind formula has a much simpler form.[10,11] Figure 1 shows Equation 8 compared to the bow shock solution from W96.[10] The line wind-driven bow shock is more open than the 3D solution from a spherical wind, when both are normalized to the standoff radius.

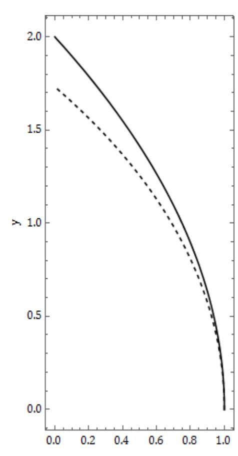

Figure 1: Comparison of the bow shock shape of W96 (dashed line) and the line wind analytic solution (solid line).

3. Colliding Winds in the Rotating Frame

We examine the problem of two identical line winds, in a counter-clockwise circular orbit about the midpoint between them (the origin). The problem is formulated assuming steady state. We likewise assume there is efficient cooling and local mixing of the winds throughout the shock structure. In steady state, fluid striking the shell must move along it and cannot build up at any point. The system has plane symmetry, and therefore we need only apply this formalism to the orbital plane (or equivalently, a thin slice bounded by two planes, just above and below the midplane).

In order to describe the shock structure with orbital motion we include non-inertial forces in our consideration of the problem. The Coriolis and centrifugal forces are of pivotal importance in properly describing the shape of the shock, as well as the pre-shock streamlines. These forces are primarily responsible for the spiral patterns observed in binary star systems.[2]

3.1. Pre-shock winds

Imagine, in the inertial frame, that one fires a projectile pointed straight toward the other star. The projectile will follow a straight line, but by the time it gets to where the second star used to be, that star will have moved out of the way. Additionally, the projectile was launched from a moving platform, and has a sideways component of motion relative to the desired target. In the corotating frame, these effects are taken into account in the non-inertial terms (Coriolis and centrifugal forces), resulting in curved streamlines.

In the reference frame of the line winds, known as the co-rotating frame, there exists a steady-state solution for the streamline trajectories and velocities of the emitted winds. The path of a fluid element launched at time τ from the star's location is described by Wilkin & Hausner (2017), hereafter WH17.[12] The equations of WH17 take forms suitable for a line wind when the azimuthal angle $\alpha = \theta + \pi$ and the latitude $\delta = 0$. The first line wind is located at $\vec{r}_* = (R_o, 0)$ and rotates with an angular frequency ω. As defined in WH17, the dimensionless trajectory, in the co-rotating frame, of a fluid element from a line wind launched at the angle $\theta_1 = 0$ is,

$$\begin{aligned}\vec{r}_1 = \langle x_1, y_1 \rangle =& \langle \left(1 - \frac{\varphi_1}{p}\right) \cos\varphi_1 + \varphi_1 \ \sin\varphi_1, \\ & \left(\frac{\varphi_1}{p} - 1\right) \sin\varphi_1 + \varphi_1 \ \cos\varphi_1 \rangle,\end{aligned} \tag{9}$$

where p is the ratio of the orbital speed of the line winds to the wind speeds, $p = \omega R_o/V_w$. The angle θ_1 is measured counter clockwise relative to the $-x$-axis for the first wind. The dimensionless parameter φ_1 represents the angle in radians that the line wind has rotated along its orbit since the launch of the fluid element in question. We also define a coordinate along a streamline, s_1, which is the time since launch, $(t - \tau_1)R_o/V_w$, of a specific streamline to a point on the streamline, where τ_1 is the time of launch. There are similar parameters for the second wind that are written as φ_2, s_2, and θ_2.

The streamline trajectories for both winds are shown in Figure 2 for $p = 0.6$. Also shown is the trajectory given by Equation 9 as the black streamline from the first wind. This streamline illustrates the need to include orbital motion in the pre-shock wind. It is clear from the figure that the streamlines are not radial as there distance from the star increases, though they are emitted radially, due to the Coriolis and centrifugal forces in the co-rotating frame. In the inertial frame, fluid emitted from the winds moves along straight lines, but these lines point back to where the source was at launch time, rather than to the current location of the launching star (line).

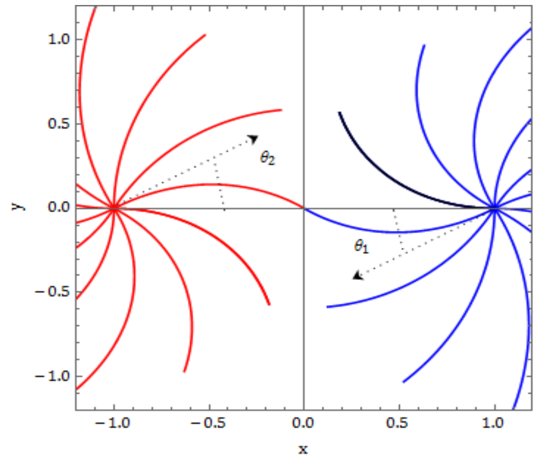

Figure 2: Stellar wind streamlines in the orbital plane from winds at $x/R_o = -1$ and $x/R_o = 1$ for $p = 0.6$. This plot depicts the skew symmetric nature of the configuration. The launch angles θ_1 and θ_2 are measured from their respective source. The black streamline represents the trajectory for the launch angle $\theta_1 = 0$.

Following the formalism of WH17, it is possible to develop a new solution for the density of an orbiting line wind. This requires the use of tensor calculus, and an inner boundary condition that the density satisfies Equation 1 close to the launch line. The complete description of the pre-shock winds also requires similar equations for the trajectory, velocity, and density of a second line wind located at $\vec{r}_2 = (-R_o, 0)$. The equations describing the second line wind develop from the inherent skew symmetry of the two-wind system in the co-rotating frame. We find that $\vec{r}_2(\theta_2, s_2) = -\vec{r}_1(\theta_1, s_1)$, which can be seen in Figure 2. The azimuthal launch angle from the second wind, θ_2, is measured counter-clockwise from the top of the x-axis as seen in Figure 2. We have assumed that the winds have equal mass and momentum loss rates, as well as equal wind speeds.

3.2. Conservation Equations

To solve for the shape of the shell, we formulated a set of differential equations pertaining to the spatial components of momentum, Φ_x and Φ_y, and the conservation of mass Φ_m within the shell. We used the approach described WS2003 to derive these equations.[13] For the remainder of this paper, we refer to the set of differential equations as conservation equations, but it should be noted that the momentum equations are not true conservation equations in the co-rotating frame due to addition of Coriolis and centrifugal forces. The equation for mass is a true conservation equation regardless of reference frame. The Coriolis and centrifugal forces may be written as,

$$F_{\text{coriolis}} = -2\sigma_x \vec{\omega} \times \vec{u}, \tag{10a}$$
$$F_{\text{centrifugal}} = -\sigma_x \vec{\omega} \times (\vec{\omega} \times \vec{r}), \tag{10b}$$

where $\sigma_x = \rho \Delta x$ is the mass column density in the $\hat{x}$-direction. These forces appear in the differential equations in component form and disappear for no orbital motion. We write a differential equation for the shape of the shell, in the method of W96 Equation 8, referred to as the trajectory equation.[10] The momentum, mass, and trajectory equations comprise a set of four ordinary differential equations (ODEs) in four dependent variables x, Φ_m, Φ_x, and Φ_y with independent variable y. The full set of differential equations will be published elsewhere. The equations are ordinary *i.e.* no partial derivatives appear due to the assumption of plane symmetry and steady state in the co-rotating frame.

It is convenient to convert all variables in the differential equations to nondimensional forms. This allows the use of the nondimensional velocities taken from WH17. The dimensionless set of differential equations are written in terms of dependent variables and known velocities which are in turn dependent on the dimensionless parameters s, θ, and p. For instance, the dimensionless mass column density $\tilde{\sigma}_x$ can be written as $\tilde{\sigma}_x = \tilde{\Phi}_m^2/\tilde{\Phi}_y$.

3.3. Singular Points and Singular Differential Equations

Here we introduce the definition of a singular differential equation, followed by a simple example. For a given differential equation, a singular point is defined as the point at which no analytic solution can be obtained due to the fact that there is the chance that when separating variables we divide by zero. A solution to a differential equation is called a singular solution when the solution is not unique at a specific point. In most cases there is not an analytic solution to a singular differential equation, and the solution is most probably discontinuous.

Consider the first-order autonomous differential equation with the initial condition $x(0) = 0$,

$$\frac{dx}{dt} = x^2. \tag{11}$$

Separating variables we find,

$$\frac{dx}{x^2} = dt. \tag{12}$$

Integrating the differential equation yields,

$$\frac{-1}{x} = t + C \tag{13}$$

and solving for $x(t)$ returns the general solution,

$$x(t) = \frac{-1}{t + C}. \tag{14}$$

The initial condition requires that $x(0) = 0$, but in solving for the particular solution we find that there is no value of the constant C that fits the initial condition. Therefore, $x_o = 0$ is a singular point and $x = x_o = 0$ is a singular solution to the differential equation. The singular solution must be noted in conjunction with the general solution to represent the full solution to the differential equation.

3.4. Stagnation Point Expansion

The solution to the locus of the thin shell requires integration of the differential equations. We will begin our integration at the origin, which for identical winds is not only the midpoint between the stars, but the stagnation point. At this location, the winds have balancing ram pressures. At the stagnation point we require that the fluxes Φ_m, Φ_x, and Φ_y are zero. This is because at the location of the stagnation point (origin) the velocity of fluid elements within the shell begins at zero. This produces the problem of a singular point at the origin. It is the singularity that leads to the often cited condition of ram pressure balance at the stagnation point. Since other points along the shell are not singular points, it is inappropriate to think of ram pressure balance at locations where fluid has non-zero speed and follows a curved path. The singularity is produced by the vanishing denominators in those equations at the stagnation point. To escape the neighborhood of the singular point, we produce Taylor series expansions in p and y for the differential equations around the stagnation point to find approximate solutions to the equations. We solve for self-consistent values that serve as the initial conditions for our numerical integration out of the singularity. The approximation of small values of p is equivalent to considering the limit of wide binaries, and is useful in making the expansions tractable.

4. Results

We integrated using the initial conditions to determine the locus of the thin-shell for winds with equal momentum loss rates and wind speeds. Numerical integration in Wolfram Mathematica yields the family of curves represented in Figure 3 for small values of the parameter p. The orbital motion of the system distorts the shock from the expected shell shape for no orbital motion, which is a vertical line midway between the stars. For systems with larger orbital speeds, the Coriolis and centrifugal forces increase proportional to pu and p^2r respectively, where u is the velocity of a fluid element in the shell and r is a distance from the origin. The non-inertial Coriolis and centrifugal effects twist the shell as seen in observations of colliding-wind binaries.[2]

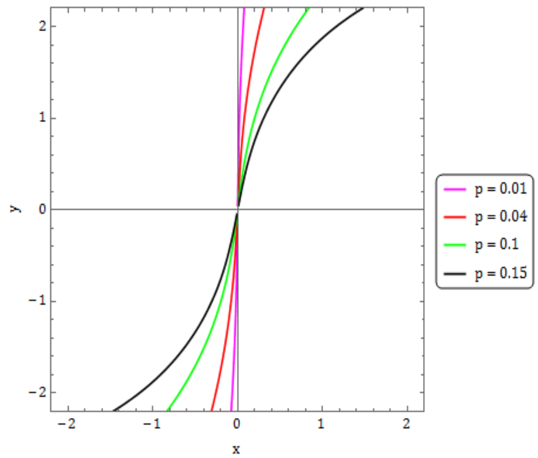

Figure 3: Numerical integration of the differential equations yields the shell shape for multiple values of p, ranging from 0.01 to 0.15. Identical winds are assumed.

5. Conclusions

We have described the concept of a line wind to create a system geometry amenable to analytic modeling. A line wind launches gas from a vertical line in initially radial directions with cylindrical and plane symmetry. In systems with orbital motion, the fluid's path is curved by the effects of both Coriolis and centrifugal forces breaking the initial cylindrical symmetry. In this paper, we used the line wind to simplify the geometry for two collision problems. A bow shock is developed by a star's supersonic movement through space with respect to the interstellar medium. We have solved for the shape of line wind bow shocks. In a binary star system, in the presence of orbital motion, a shocked structure developed by the collision of two stellar winds becomes distorted into a spiral structure. We have presented initial results for the shape of this structure for a two line wind collision.

First, we applied the approximation of a line wind to the steady-state bow shock due to a wind source moving with respect to a uniform medium. Using the flux function formalism of Cantó, Raga, & Wilkin (1996) we solved for the shape of the bow shock. We have found that line wind bow shocks have a wider opening angle compared to the bow shock created by an isotropic wind found by Wilkin (1996).

Secondly, we formulated the binary line wind collision problem in the co-rotating frame including Coriolis and centrifugal forces, assuming equal masses and circular orbits for the two stars. The set of four differential equations is singular at the origin, and therefore required Taylor series expansions to solve for initial conditions. The expansions and integrations were completed for stars with equal wind speeds and equal momentum loss rates. There is an increasing distortion in shell shape with increasing values of p, which corresponds to greater orbital speed relative to wind speed. The initial results of our solution to this problem are published in this paper. A more general and in-depth presentation of the binary line wind collision problem will be published in a future paper.[14]

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